



# Characterization of double domination subdivision number of trees

M. Atapour<sup>a</sup>, Abdollah Khodkar<sup>b</sup>, S.M. Sheikholeslami<sup>c, 1</sup>

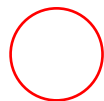
<sup>a</sup>Department of Mathematics, Alzahra University, Vanak Square 19834, Tehran, Islamic Republic of Iran

<sup>b</sup>Department of Mathematics, University of West Georgia, Carrollton, GA 30118, USA

<sup>c</sup>Department of Mathematics, Azarbaijan University of Tarbiat Moallem, Tabriz, Islamic Republic of Iran

Received 2 November 2006; received in revised form 14 March 2007; accepted 15 March 2007

Available online 23 March 2007



## Abstract

In a graph  $G$ , a vertex *dominates* itself and its neighbors. A subset  $S \subseteq V(G)$  is a *double dominating set* of  $G$  if  $S$  dominates every vertex of  $G$  at least twice. The *double domination number*  $dd(G)$  is the minimum cardinality of a double dominating set of  $G$ . The *double domination subdivision number*  $sd_{dd}(G)$  is the minimum number of edges that must be subdivided (each edge in  $G$  can be subdivided at most once) in order to increase the double domination number. In this paper first we establish upper bounds on the double domination subdivision number for arbitrary graphs in terms of vertex degree. Then we present several different conditions on  $G$  which are sufficient to imply that  $sd_{dd}(G) \leq 3$ . We also prove that  $1 \leq sd_{dd}(T) \leq 2$  for every tree  $T$ , and characterize the trees  $T$  for which  $sd_{dd}(T) = 2$ .

© 2007 Elsevier B.V. All rights reserved.

MSC: 05C69

Keywords: Double domination number; Double domination subdivision number

## 1. Introduction

In this paper,  $G$  is a simple connected graph with vertex set  $V(G)$  and edge set  $E(G)$  (briefly  $V$  and  $E$ ). For every vertex  $v \in V$ , the *open neighborhood*  $N(v)$  is the set  $\{u \in V(G) \mid uv \in E(G)\}$  and the *closed neighborhood* is the set  $N[v] = N(v) \cup \{v\}$ . The *open neighborhood* of a set  $S \subseteq V$  is the set  $N(S) = \bigcup_{v \in S} N(v)$ , and the *closed neighborhood* of  $S$  is the set  $N[S] = N(S) \cup S$ .

A vertex  $v \in V$  *dominates* itself and its neighbors. A subset  $S$  of vertices of  $G$  is a *dominating set* if  $N[S] = V$  (that is,  $S$  dominates  $V$ ). The *domination number*  $\gamma(G)$  is the minimum cardinality of a dominating set of  $G$ , and a dominating set of minimum cardinality is called a  $\gamma$ -set [8]. A subset  $S$  of  $V$  is a *double dominating set* of  $G$  if  $S$  dominates every vertex of  $G$  at least twice [5]. Note that a double domination set of  $G$  is defined only if  $G$  has no isolated vertices. The *double domination number*  $dd(G)$  is the minimum cardinality of a double dominating set of  $G$ . A  $dd(G)$ -set is a double dominating set of  $G$  with cardinality  $dd(G)$ . Throughout this paper when we talk about  $dd(G)$  we assume that  $G$  has no isolated vertices. For a more thorough treatment of domination parameters and for terminology not presented here, see [8,12].

<sup>1</sup> Research supported by the Research Office of Azarbaijan University of Tarbiat Moallem.

E-mail addresses: [akhodkar@westga.edu](mailto:akhodkar@westga.edu) (A. Khodkar), [s.m.sheikholeslami@azaruniv.edu](mailto:s.m.sheikholeslami@azaruniv.edu) (S.M. Sheikholeslami).

The (total) domination subdivision number ( $sd_{\gamma_t}(G)$ )  $sd_{\gamma_t}(G)$  of a graph  $G$  is the minimum number of edges that must be subdivided (where each edge in  $G$  can be subdivided at most once) in order to increase the (total) domination number of  $G$ , see [6,7]. (An edge  $uv \in E(G)$  is subdivided if the edge  $uv$  is deleted, but a new vertex  $x$  is added, along with two new edges  $ux$  and  $vx$ .) The domination subdivision numbers and the total domination subdivision numbers have been studied by several authors (see for example [2,4,6,7,9–11]). In particular, in [7] it is proved that  $1 \leq sd_{\gamma_t}(T) \leq 3$ , where  $sd_{\gamma_t}(T)$  is the total domination subdivision number of tree  $T$  with at least three vertices. Moreover, the authors of [1,9] give a constructive characterization of trees whose domination subdivision number and total domination subdivision number is 3, respectively.

The purpose of this paper is to initialize the study of the double domination subdivision number  $sd_{dd}(G)$ . That is, the double domination subdivision number  $sd_{dd}(G)$  of a graph  $G$  is the minimum number of edges that must be subdivided (where each edge in  $G$  can be subdivided at most once) in order to increase the double domination number of  $G$ . Although it may not be immediately obvious that the double domination subdivision number is defined for all connected graphs of order  $n \geq 2$ , we will show this shortly.

We make use of the following results in this paper.

**Theorem A** (Harary and Hages [5]). For  $n \geq 2$ ,  $dd(P_n) = \lceil (2n + 2)/3 \rceil$ .

**Theorem B** (Harary and Hages [5]). Let  $G$  be a graph of order  $n$ . Then  $dd(G) = 2$  if and only if there exists vertices  $u, v \in V(G)$  such that  $\deg(u) = \deg(v) = n - 1$ .

Recall that a matching in a graph is a set of non-loop edges with no shared endpoints. The maximum size of a matching in  $G$  is denoted by  $\alpha'(G)$ .

**Theorem C** (Brandt [3]). Let  $G$  be a simple graph of order  $n$  such that  $\delta \geq k$  and  $n \geq 2k$  for some  $k \in \mathbb{N}$ . Then  $\alpha'(G) \geq k$ .

## 2. Bounds on the double domination subdivision number

In this section we present some upper bounds on  $sd_{dd}(G)$  in terms of the vertex degree and the minimum degree of  $G$ . We begin with three propositions giving some sufficient conditions for a graph to have a small double domination subdivision number.

**Proposition 1.** For any graph  $G$  of order  $n \geq 2$  and  $dd(G) = 2$ ,  $sd_{dd}(G) = 1$ .

**Proof.** For  $n = 2$  the statement is trivial. Now let  $n \geq 3$ . Then the graph  $G'$ , obtained by subdividing any edge of  $G$ , has no vertex of degree  $n(G') - 1$ . Therefore,  $dd(G') > 2 = dd(G)$ , by Theorem B. This implies that  $sd_{dd}(G) = 1$ .  $\square$

**Proposition 2.** For any graph  $G$  with  $dd(G) = 3$ ,  $sd_{dd}(G) \leq 2$ .

**Proof.** Let  $S = \{v_1, v_2, v_3\}$  be a  $dd$ -set of  $G$ . Obviously, the induced subgraph  $G[S]$  is connected. We may assume  $v_1v_2, v_2v_3 \in E(G)$ . Let  $G'$  be obtained from  $G$  by subdividing the edges  $v_1v_2, v_2v_3$  with new vertices  $x_1, x_2$ , respectively. Let  $S'$  be a  $dd(G')$ -set. Clearly,  $|S' \cap \{v_i, x_i, v_{i+1}\}| \geq 2$  for  $i = 1, 2$ . If  $x_1, x_2 \notin S'$ , then  $\{v_1, v_2, v_3\} \subseteq S'$ . Now in order to dominate  $v_2$  twice we have  $|S'| \geq 4$ . We may assume  $x_1 \in S'$ . If  $x_2 \notin S'$ , then  $v_2, v_3 \in S'$  and since  $v_3$  must be dominated twice, we have  $|S'| \geq 4$ . Let  $x_2 \in S'$ . If  $v_2 \notin S'$ , then  $v_1, v_3 \in S'$  and so  $|S'| \geq 4$ . Let  $v_2 \in S'$ . Now  $v_1$  must be dominated twice. This forces  $|S'| \geq 4$  and the result follows.  $\square$

**Proposition 3.** If  $G$  contains a strong support vertex, then  $sd_{dd}(G) = 1$ .

**Proof.** Let  $u, v \in V(G)$  be two leaves adjacent to  $w$  and let  $G'$  be obtained from  $G$  by subdividing the edge  $uw$  with vertex  $x$ . Let  $S$  be a  $dd(G')$ -set. Obviously,  $\{u, v, x, w\} \subseteq S$ . Now since  $S \setminus \{x\}$  is a double dominating set in  $G$  it follows that  $sd_{dd}(G) = 1$ .  $\square$

**Theorem 4.** For any graph  $G$  of order  $n \geq 2$  and  $\delta = 1$ ,  $sd_{dd}(G) \leq 2$ .

**Proof.** Let  $v \in V$  be a vertex of degree 1 and assume  $u$  is adjacent to  $v$ . If  $\deg(u) = 1$ , then  $sd_{dd}(G) = 1$ . So, assume  $w \neq v$  is adjacent to  $u$ . Let  $G'$  be obtained from  $G$  by subdividing the edges  $uv$  and  $uw$  with vertices  $x$  and  $y$ , respectively, and let  $S$  be a  $dd(G')$ -set. Obviously,  $v, x \in S$  and  $|S \cap \{u, y, w\}| \geq 2$ . Now if  $y, w \in S$ , then  $(S \setminus \{x, y\}) \cup \{u\}$  is a double dominating set in  $G$ . The cases  $u, y \in S$  or  $u, w \in S$  are similar. Hence,  $sd_{dd}(G) \leq 2$ .  $\square$

An immediate consequence of Theorem 4 is:

**Corollary 5.** For any nontrivial tree  $T$ ,  $sd_{dd}(T) \leq 2$ .

**Theorem 6.** For every simple connected graph  $G$  with  $\delta \geq 2$ ,  $sd_{dd}(G) \leq \delta$ .

**Proof.** Let  $v \in V(G)$  be a vertex of degree  $\delta$  and let  $N(v) = \{v_1, v_2, \dots, v_\delta\}$ . Suppose that  $G'$  is obtained from  $G$  by subdividing the edges  $vv_1, vv_2, \dots, vv_\delta$  with vertices  $x_1, x_2, \dots, x_\delta$ , respectively. Let  $S$  be a  $dd(G')$ -set. Then  $|S \cap \{x_1, \dots, x_\delta\}| \geq 1$ . If  $v \in S$ , then we may assume  $x_1 \in S$ . We note that in order to dominate  $x_i$  twice we must have  $|\{x_i, v_i\} \cap S| \geq 1$  for  $i \in \{1, 2, 3, \dots, \delta\}$ . Hence,  $(S \setminus \{x_1, \dots, x_\delta\}) \cup \{v_2, \dots, v_\delta\}$  is a double dominating set of  $G$  whose size is less than  $|S|$ . If  $v \notin S$ , then  $\{x_i, v_i : 1 \leq i \leq \delta\} \subseteq S$ . Now  $(S \setminus \{x_1, \dots, x_\delta\}) \cup \{v\}$  is a double dominating set of  $G$ . Therefore  $sd_{dd}(G) \leq \delta$ .  $\square$

A consequence of Theorems 4 and 6 is that  $sd_{dd}(G)$  is defined for every simple connected graph  $G$  of order  $n \geq 2$ .

**Theorem 7.** For any simple connected graph  $G$  with adjacent vertices  $u$  and  $v$ , each of degree at least two,

$$sd_{dd}(G) \leq \deg(u) + \deg(v) - |N(u) \cap N(v)| - 2.$$

Furthermore, this bound is sharp.

**Proof.** Let  $N(v) = \{v_1, v_2, \dots, v_k\}$ , where  $u = v_1$ , and if  $N(u) \setminus N[v] \neq \emptyset$ , let  $N(u) \setminus N[v] = \{u_1, \dots, u_t\}$ . Let  $G'$  be the graph obtained by subdividing the edge  $vv_i$  with vertex  $x_i$  for  $2 \leq i \leq k$ , and the edge  $uu_j$  with vertex  $y_j$  for  $1 \leq j \leq t$ . Let  $A = \{x_i, y_j \mid 2 \leq i \leq k \text{ and } 1 \leq j \leq t\}$  and let  $S'$  be a  $dd(G')$ -set such that  $|S' \cap A|$  is minimum. We prove that  $dd(G) \leq |S'| - 1$ .

If both  $u$  and  $v$  are in  $S'$ , then  $S' \cap A = \emptyset$  since  $|S' \cap A|$  is minimum. Hence,  $S' \setminus \{u\}$  is a double dominating set in  $G$ . Assume  $u \in S'$  and  $v \notin S'$  (the case  $u \notin S'$  and  $v \in S'$  is similar). If  $k \geq 3$ , then  $(S' \setminus \{x_2, x_3\}) \cup \{v\}$  is a double dominating set for  $G$ . If  $k = 2$  and  $t \geq 1$ , then  $(S' \setminus \{x_2, y_1\}) \cup \{v\}$  is a double dominating set for  $G$ . If  $k = 2$  and  $t = 0$ , then  $S' \setminus \{x_2\}$  is a double dominating set for  $G$ . Now let  $u, v \notin S'$ . Then  $\{x_i, v_i, y_j, u_j : 2 \leq i \leq k, 1 \leq j \leq t\} \subseteq S'$  and  $\deg(u), \deg(v) \geq 3$ , since  $u$  and  $v$  must be dominated twice by  $S$ . If  $t \geq 1$ , then  $(S' \setminus \{x_i, y_j : 2 \leq i \leq k, 1 \leq j \leq t\}) \cup \{u, v\}$  is a double dominating set for  $G$ . If  $t = 0$ , then  $(S' \setminus \{x_i : 2 \leq i \leq k\}) \cup \{u\}$  is a double dominating set for  $G$ .

Therefore  $dd(G) < dd(G')$  and  $sd_{dd}(G) \leq \deg(u) + \deg(v) - |N(u) \cap N(v)| - 2$ . For a cycle of order  $n$ ,  $n \equiv 2 \pmod{3}$ , this bound is attained. This completes the proof.  $\square$

**Lemma 8.** If  $G$  contains a matching  $M$  such that  $dd(G) \leq 2|M| - 1$ , then  $sd_{dd}(G) \leq |M|$ .

**Proof.** Let  $G'$  be obtain from  $G$  by subdividing every edge of  $M$ . Each double dominating set of  $G'$  has order at least  $2|M|$ . Hence,  $dd(G') > dd(G)$  and  $sd_{dd}(G) \leq |M|$ .  $\square$

**Theorem 9.** For every simple connected graph  $G$  of order  $n \geq 2$ ,  $sd_{dd}(G) \leq \lfloor n/2 \rfloor$ .

**Proof.** For  $n = 2$  or  $3$ , the proof is clear. Assume  $n \geq 4$ . If  $\delta \leq \lfloor n/2 \rfloor$ , the statement holds by Theorem 6. Suppose that  $\delta > \lfloor n/2 \rfloor$  which implies  $\delta \geq 3$ . If  $x, y \in V(G)$ , then obviously  $V(G) \setminus \{x, y\}$  is a double dominating set of  $G$  and so  $dd(G) \leq n - 2$ . Apply Theorem C with  $k = \lfloor n/2 \rfloor$  to see that  $\alpha'(G) \geq \lfloor n/2 \rfloor$ . Let  $M$  be a matching in  $G$  with  $\lfloor n/2 \rfloor$  edges. We have  $dd(G) \leq n - 2 \leq 2|M| - 1$ . Hence, by Theorem 8,  $sd_{dd}(G) \leq \lfloor n/2 \rfloor$ .  $\square$

For a path of order 4 or a cycle of order 5, the equality in Theorem 9 is attained.

### 3. Trees whose double domination subdivision numbers is 2

By Corollary 5, trees can be classified as Class 1 or Class 2 depending on whether their double domination subdivision number is 1 or 2, respectively. In this section we provide a constructive characterization of all trees in Class 2. For this purpose, we describe a procedure to build a family  $\mathcal{F}$  of labeled trees that are in Class 2 as follows. First we define the following operations on labeled trees. The label of a vertex is also called its *status* and denoted  $sta(v)$ . Let  $\mathcal{F}$  be the family of labeled trees that:

1. contains  $P_4$  where the two leaves have status  $C$ , and the two support vertices have status  $B$ , and
2. is closed under the two operations  $\mathfrak{T}_1, \mathfrak{T}_2$ , which extend the tree  $T$  by attaching a tree to the vertex  $y \in V(T)$ , called the *attacher*.

**Operation  $\mathfrak{T}_1$ .** Assume  $sta(y) = C$ . Then add a path  $yxwv$  to  $T$  with  $sta(x) = A$ ,  $sta(w) = B$  and  $sta(v) = C$  (see Fig. 1).

**Operation  $\mathfrak{T}_2$ .** Assume  $sta(y) \in \{A, B\}$ . Then add a path  $yxv$  to  $T$  with  $sta(x) = B$  and  $sta(v) = C$  (see Fig. 1).

### 4. The family $\mathcal{F}$

If  $T \in \mathcal{F}$ , we let  $A(T)$ ,  $B(T)$  and  $C(T)$  be the set of vertices of status  $A$ ,  $B$  and  $C$ , respectively, in  $T$ . The relation  $|B(T)| = |C(T)|$  and the following observation come from the way in which each tree in the family  $\mathcal{F}$  is constructed.

**Observation 10.** Let  $T \in \mathcal{F}$  and  $v \in V(T)$ .

1. If  $v$  is a leaf, then  $sta(v) = C$ .
2. If  $v$  is a support vertex, then  $sta(v) = B$ .
3. If  $sta(v) = A$ , then  $v$  is adjacent to exactly one vertex of status  $C$ , and at least one vertex of status  $B$ .
4. If  $sta(v) = B$ , then  $v$  is adjacent to exactly one vertex of status  $C$ .
5. If  $sta(v) = C$ , then  $v$  is adjacent to exactly one vertex, say  $x$ , of status  $B$ . Moreover,  $N(v) \setminus \{x\} \subseteq A(T)$ .
6. The distance between any two vertices in  $C(T)$  is at least three.
7. Let  $P_4 = v_1v_2v_3v_4$  with  $sta(v_1) = sta(v_4) = C$  and  $sta(v_2) = sta(v_3) = B$ . Assume  $T$  is obtained from  $P_4$  by successive operations  $\mathfrak{T}^1, \dots, \mathfrak{T}^m$ , respectively, where  $\mathfrak{T}^i \in \{\mathfrak{T}_1, \mathfrak{T}_2\}$  for  $1 \leq i \leq m$ . If  $v \in V(T)$ ,  $\deg(v) = 2$ ,  $sta(v) = B$  and  $v$  has a neighbor  $u$  in  $T$  such that  $\deg(u) = 2$  and  $sta(u) = B$ , then  $u$  and  $v$  lie on initial  $P_4$ .

In order to show that each tree in the family  $\mathcal{F}$  is in Class 2, we first present four lemmas.

**Lemma 11.** If  $T \in \mathcal{F}$  and  $T$  is obtained from  $T_0 = P_4$  by successive operations  $\mathfrak{T}^1, \dots, \mathfrak{T}^m$ , where  $\mathfrak{T}^i \in \{\mathfrak{T}_1, \mathfrak{T}_2\}$  for  $i = 1, 2, \dots, m$ , then  $C(T) \cup B(T)$  is a  $dd(T)$ -set and  $dd(T) = 2m + 4$ .

**Proof.** By Observation 10 (3, 4, 5),  $B(T) \cup C(T)$  is a double dominating set of  $T$  implying that  $dd(T) \leq |B(T)| + |C(T)| = 2|C(T)|$ . Now let  $S$  be a  $dd(T)$ -set. For each  $x \in C(T)$ ,  $|S \cap N[x]| \geq 2$  and if  $x, y \in C(T)$ , then  $N[x] \cap N[y] = \emptyset$ , by Observation 10(6). This implies  $|S| \geq 2|C(T)|$ . Therefore,  $C(T) \cup B(T)$  is a  $dd(T)$ -set. Since  $|B(P_4) \cup C(P_4)| = 4$  and each operation  $\mathfrak{T}_1$  and  $\mathfrak{T}_2$  adds two more vertices of  $C(T) \cup B(T)$ , it follows that  $|B(T) \cup C(T)| = 2m + 4$ .  $\square$

**Lemma 12.** Let  $T \in \mathcal{F}$  and  $v \in C(T)$ . Then there exists a set  $S$  containing  $v$  that doubly dominates  $V(T) \setminus \{v\}$  and  $|S| = dd(T) - 1$ .

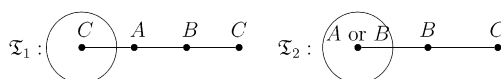


Fig. 1. The two operations.

**Proof.** Let  $P_4 = v_1 v_2 v_3 v_4$  and let  $T$  be obtained from  $P_4$  by successive operations  $\mathfrak{T}^1, \dots, \mathfrak{T}^m$ , respectively, where  $\mathfrak{T}^i \in \{\mathfrak{T}_1, \mathfrak{T}_2\}$  for  $1 \leq i \leq m$  if  $m \geq 1$  and  $T = P_4$  if  $m = 0$ . The proof is by induction on  $m$ . If  $m = 0$ , then clearly the statement is true. Assume  $m \geq 1$  and that the statement holds for all trees which are obtained from  $P_4$  by applying at most  $m - 1$  operations  $\mathfrak{T} \in \{\mathfrak{T}_1, \mathfrak{T}_2\}$ . Let  $T$  be rooted at  $v_2$ . Assume  $u_1 \notin \{v_1, v_4\}$  is a leaf with maximum distance from  $v_2$  and let  $u_2$  be the parent of  $u_1$ . By Observation 10(6),  $d(u_1, v_2) \geq 2$ . If  $d(u_1, v_2) = 2$ , then we must have  $\mathfrak{T}^i = \mathfrak{T}_2$  for each  $1 \leq i \leq m$ . Let  $T_{m-1} = T \setminus \{u_1, u_2\}$ . By inductive hypothesis, for every vertex  $u \in C(T_{m-1})$ , there exists a set  $S_u$  containing  $u$  that doubly dominates  $V(T) \setminus \{u\}$  and  $|S_u| = dd(T_{m-1}) - 1$ . Take  $S = S_u \cup \{u_1, u_2\}$  if  $u \in C(T_{m-1})$  and  $S = S_{v_1} \cup \{v_2, u_1\}$  if  $u = u_1$ . Then  $S$  is the required set.

Now let  $d(v_2, u_1) \geq 3$  and let  $u_3$  be the parent of  $u_2$ . By Observation 10(6),  $\deg(u_2) = 2$  and  $sta(u_3) = A$  or  $B$ . First let  $\deg(u_3) = 2$  and assume  $u_4$  is the parent of  $u_3$ . Obviously,  $sta(u_3) = A$  and  $sta(u_4) = C$ . Now we may assume  $\mathfrak{T}^m = \mathfrak{T}_1$  which adds the path  $u_4 u_3 u_2 u_1$ . Let  $T_{m-1} = T \setminus \{u_1, u_2, u_3\}$ . By inductive hypothesis, for any vertex  $u \in C(T_{m-1})$ , there exists a set  $S_u$  containing  $u$  that doubly dominates  $V(T_{m-1}) \setminus \{u\}$  and  $|S_u| = dd(T_{m-1}) - 1$ . Take  $S = S_u \cup \{u_1, u_2\}$  if  $u \in C(T_{m-1})$  and  $S = S_{u_4} \cup \{u_3, u_1\}$  if  $u = u_1$ . Then  $S$  is the required set.

Now let  $\deg(u_3) \geq 3$ . Then  $u_3$  is a support vertex or is adjacent to a support vertex. By Observation 10(4),  $sta(u_3) = A$  or  $B$ . So, we can assume  $\mathfrak{T}^m = \mathfrak{T}_2$  which adds the path  $u_3 u_2 u_1$ . Let  $T_{m-1} = T \setminus \{u_1, u_2\}$ . By inductive hypothesis, for any vertex  $u \in C(T_{m-1})$ , there exists a set  $S_u$  containing  $u$  that doubly dominates  $V(T) \setminus \{u\}$  and  $|S_u| = dd(T_{m-1}) - 1$ . Take  $S = S_u \cup \{u_1, u_2\}$  if  $u \in C(T_{m-1})$ ,  $S = B(T_{m-1}) \cup C(T_{m-1}) \cup \{u_1\}$  if  $u = u_1$ ,  $sta(u_3) = B$  and  $S = S_x \cup \{u_3, u_1\}$ , where  $x \in N(u_3)$  and  $sta(x) = C$  if  $u = u_1$  and  $sta(u_3) = A$ . Then  $S$  is the required set.  $\square$

**Lemma 13.** *If  $T \in \mathcal{F}$ , then every vertex of  $V(T)$  is in some  $dd(T)$ -set.*

**Proof.** If  $v \in B(T) \cup C(T)$ , then the statement holds by Lemma 11. Now let  $v \in A(T)$ . By Observation 10(3),  $v$  is adjacent to a vertex  $u \in C(T)$ . By Lemma 12, there exists a set  $S$  containing  $u$  that doubly dominates  $V(T) \setminus \{u\}$  and  $|S| = dd(T) - 1$ . Now  $S \cup \{v\}$  is the required set.  $\square$

**Lemma 14.** *Let  $T \in \mathcal{F}$  and let  $T^*$  be a tree obtained from  $T$  by subdividing one edge of  $T$ . Then  $dd(T^*) = dd(T)$ .*

**Proof.** Let  $T \in \mathcal{F}$ . First note that  $dd(T^*) \geq dd(T)$  and that any double dominating set of  $T^*$  of order  $dd(T)$  is a  $dd(T^*)$ -set. Let  $e \in E(T)$  and let  $T^*$  be obtained from  $T$  by adding a new vertex  $x$  which subdivides the edge  $e$ . Let  $P_4 = v_1 v_2 v_3 v_4$  and let  $T$  be obtained from  $P_4$  by successive operations  $\mathfrak{T}^1, \dots, \mathfrak{T}^m$ , respectively, where  $\mathfrak{T}^i \in \{\mathfrak{T}_1, \mathfrak{T}_2\}$  for  $1 \leq i \leq m$  if  $m \geq 1$  and  $T = P_4$  if  $m = 0$ . The proof is by induction on  $m$ . If  $m = 0$ , then clearly the statement is true by Theorem A. Assume  $m \geq 1$  and that the statement holds for all trees which are obtained from  $P_4$  by applying at most  $m - 1$  operations. Suppose  $T_{m-1}$  is a tree obtained by applying the first  $m - 1$  operations  $\mathfrak{T}^1, \dots, \mathfrak{T}^{m-1}$ . When  $e \in E(T_{m-1})$ , let  $T_{m-1}^*$  be obtained from  $T_{m-1}$  by subdividing the edge  $e$ . Let  $T$  be rooted at  $v_2$ . Assume  $u_1 \notin \{v_1, v_4\}$  is a leaf with maximum distance from  $v_2$  and  $u_2$  is the parent of  $u_1$ . By Observation 10(6),  $d(u_1, v_2) \geq 2$ . If  $d(u_1, v_2) = 2$ , then we must have  $\mathfrak{T}^i = \mathfrak{T}_2$  for each  $1 \leq i \leq m$ . We may assume  $T_{m-1} = T \setminus \{u_1, u_2\}$ . If  $e \in E(T_{m-1})$ , then by inductive hypothesis,  $dd(T_{m-1}^*) = dd(T_{m-1})$ . Also, we can easily check that  $dd(T^*) \leq dd(T_{m-1}^*) + 2 = dd(T)$ . Hence,  $dd(T^*) = dd(T)$  and the result follows.

Now assume  $e = v_2 u_2$  (or  $e = u_1 u_2$ ) and let  $S$  be a  $dd(T_{m-1})$ -set. Since  $v_2$  is a support vertex,  $v_2 \in S$ . Then  $S \cup \{u_1, u_2\}$  (respectively,  $S \cup \{u_1, x\}$ ) is a double dominating set of  $T^*$ . Therefore,  $dd(T^*) \leq dd(T_{m-1}) + 2 = dd(T)$  and the result follows.

Now let  $d(v_2, u_1) \geq 3$  and assume  $u_3$  is the parent of  $u_2$ . By Observation 10(1, 5) and the fact that  $u_1$  has maximum distance from  $v_2$  we have  $\deg(u_2) = 2$  and  $sta(u_3) = A$  or  $B$ .

*Case 1:*  $\deg(u_3) = 2$ . By Observation 10(7, 4) we have  $sta(u_3) = A$  and  $sta(u_4) = C$ . Hence, we may assume  $\mathfrak{T}^m = \mathfrak{T}_1$  which adds the path  $u_4 u_3 u_2 u_1$ , where  $u_4$  is the parent of  $u_3$ . Let  $T_{m-1} = T \setminus \{u_1, u_2, u_3\}$ .

*Subcase 1.1:*  $e \in E(T_{m-1})$ . By inductive hypothesis,  $dd(T_{m-1}^*) = dd(T_{m-1})$ . If  $\deg(v_4) = 2$ , then for any  $dd(T_{m-1}^*)$ -set  $S$ ,  $S \cup \{v_1, v_2\}$  is a double dominating set for  $T^*$ . Hence,  $dd(T^*) \leq dd(T)$ . Let  $\deg(v_4) \geq 3$ . Since  $sta(u_4) = C$ ,  $u_4$  neither is a support vertex nor is adjacent to a support vertex. Hence,  $T_{v_4}$  is a star  $K_{1,t}$  ( $t \geq 2$ ) whose each edge is subdivided twice. Now we may assume  $T_1$  has a path  $v_4 x_1 x_2 x_3$  where  $x_1 \notin \{v_3, v_5\}$ . Let  $S$  be a  $dd(T_{m-1}^*)$ -set. We may assume  $u_4 \in S$ . Then  $S \cup \{u_1, u_2\}$  is a double dominating set for  $T^*$ . Hence,  $dd(T^*) \leq dd(T)$  and the result follows.

*Subcase 1.2:*  $e \in E(T) \setminus E(T_{m-1})$ . Let  $e = u_2 u_3$  ( $e = u_1 u_2$ ). By Lemma 12, there exists a set  $S$  containing  $u_4$  that doubly dominates  $V(T_{m-1}) \setminus \{u_4\}$  and  $|S| = dd(T_{m-1}) - 1$ . Then  $S \cup \{u_1, u_2, u_3\}$  ( $S \cup \{u_1, x, u_3\}$ ) is a double dominating set for  $T^*$ . Hence,  $dd(T^*) \leq dd(T)$  and the result follows.

Case 2:  $\deg(u_3) \geq 3$ . Then  $u_3$  is a support vertex or adjacent to a support vertex.

Subcase 2.1:  $u_3$  is a support vertex. Then  $sta(u_3) = B$ . Let  $T_{m-1} = T \setminus \{u_1, u_2\}$ . If  $e \in E(T_{m-1})$ , then by inductive hypothesis  $dd(T_{m-1}^*) = dd(T_{m-1})$ . Let  $S$  be a  $dd(T_{m-1}^*)$ -set. Then  $S \cup \{u_1, u_2\}$  is a double dominating set in  $T^*$  and  $dd(T^*) \leq dd(T)$ . When  $e = u_2u_3$  or  $e = u_1u_2$  we let  $S$  be a  $dd(T_{m-1})$ -set. Then  $S \cup \{u_1, u_2\}$  or  $S \cup \{u_1, x\}$  is a double dominating set in  $T^*$ , respectively. Hence,  $dd(T^*) \leq dd(T)$ .

Subcase 2.2:  $u_3$  is adjacent to a support vertex. Then  $sta(u_3) = A$  or  $B$ . Let  $T_{m-1} = T \setminus \{u_1, u_2\}$ . If  $e \in E(T_{m-1})$ , then by inductive hypothesis  $dd(T_{m-1}^*) = dd(T_{m-1})$ . Let  $S$  be a  $dd(T_{m-1}^*)$ -set. Then  $S \cup \{u_1, u_2\}$  is a double dominating set for  $T^*$ . Hence,  $dd(T^*) \leq dd(T_{m-1}^*) + 2 = dd(T)$ . Let now  $e = u_2u_3$  ( $e = u_1u_2$ ) be subdivided with vertex  $x$ . By Observation 10(3, 4),  $u_3$  is adjacent to a vertex  $y \in C(T_{m-1})$ . By Lemma 12, there exists a set  $S$  containing  $y$  that doubly dominates  $V(T_{m-1}) \setminus \{y\}$  and  $|S| = dd(T_{m-1}) - 1$ . Now  $S \cup \{u_1, u_2, u_3\}$  ( $S \cup \{u_1, x, u_3\}$ ) is a double dominating set for  $T^*$ . So  $dd(T^*) \leq dd(T_{m-1}) + 2 = dd(T)$ . This completes the proof.  $\square$

**Theorem 15.** Each tree in Family  $\mathcal{F}$  is in Class 2.

**Proof.** If  $T \in \mathcal{F}$ , then  $sd_{dd}(T) \geq 2$  by Lemma 14. Now the result follows by Corollary 5.  $\square$

The following result is similar to Lemma 13 of [9]. The proof of this lemma is straightforward and therefore omitted.

**Lemma 16.** If  $T$  is a tree obtained from a tree  $T'$  of order at least two by adding a subdivided star  $SK_{1,t}$  ( $t \geq 2$ ) and an edge which joins the center of the star to a vertex of  $T'$ , then  $dd(T) = dd(T') + 2t$ . Moreover,  $sd_{dd}(T) \leq sd_{dd}(T')$ .

Now we are ready to prove the main theorem of this section.

**Theorem 17.** A tree  $T$  of order  $n \geq 4$  is in Class 2 if and only if  $T \in \mathcal{F}$ .

**Proof.** By Theorem 15, we only need to prove that every tree in Class 2 is in  $\mathcal{F}$ . We prove this by induction on the order  $n$  of the tree. Let  $n = 4$ . The only tree of order 4 and  $sd_{dd}(T) = 2$ , is  $P_4 \in \mathcal{F}$ . Let  $n \geq 5$  and assume the statement holds for every tree in Class 2 of order less than  $n$ . Let  $T$  be a tree of order  $n$  and  $sd_{dd}(T) = 2$ . Let  $P = v_1v_2 \dots v_r$  be a longest path in  $T$ . Obviously,  $\deg(v_1) = \deg(v_r) = 1$  and  $\deg(v_2) = \deg(v_{r-1}) = 2$  by Proposition 3. If  $r = 3$ , then obviously  $v_2$  is a strong support vertex and hence,  $sd_{dd}(T) = 1$  by Proposition 3, a contradiction. If  $r = 4$ , then either  $n = 4$  and  $T = P_4 \in \mathcal{F}$  or  $n \geq 5$  and  $sd_{dd}(T) = 1$ , a contradiction. Now let  $r \geq 5$ .

Case 1:  $\deg(v_3) = 2$ . Let  $T_1 = T \setminus \{v_1, v_2, v_3\}$ .

Subcase 1.1:  $\deg(v_4) = 2$ . Let  $S$  be a  $dd(T)$ -set. Obviously,  $v_1, v_2 \in S$  and we may assume  $v_4 \in S$ . Now if  $v_3 \in S$ , then  $S \setminus \{v_1, v_2, v_3\} \cup \{v_5\}$  is a double dominating set for  $T_1$  and if  $v_3 \notin S$ , then  $S \setminus \{v_1, v_2\}$  is a double dominating set for  $T_1$ . Therefore  $dd(T_1) \leq dd(T) - 2$ . Now if  $S_1$  is a  $dd(T_1)$ -set, then  $S_1 \cup \{v_1, v_2\}$  is a double dominating set for  $T$ . Hence,  $dd(T) = dd(T_1) + 2$ .

**Claim 1.**  $sd_{dd}(T_1) = 2$ .

**Proof of Claim 1.** Let  $e \in E(T_1)$  and  $T_1^*$  (respectively,  $T^*$ ) be obtained from  $T_1$  (respectively,  $T$ ) by subdividing the edge  $e$ . By assumption,  $dd(T^*) = dd(T)$ . If  $S$  is a  $dd(T_1^*)$ -set, then  $S \cup \{v_1, v_2\}$  is a double dominating set in  $T^*$  and so  $dd(T^*) \leq dd(T_1^*) + 2$ . Now let  $S$  be a  $dd(T^*)$ -set. Obviously,  $v_1, v_2 \in S$  and we may assume  $v_4 \in S$ . Now if  $v_3 \in S$ , then  $S \setminus \{v_1, v_2, v_3\} \cup \{x\}$ , where  $x \in N(v_4) \setminus \{v_3\}$ , is a double dominating set for  $T_1^*$  and if  $v_3 \notin S$ , then  $S \setminus \{v_1, v_2\}$  is a double dominating set for  $T_1^*$ . Hence,  $dd(T^*) = dd(T_1^*) + 2$ . Now we have  $dd(T_1) + 2 = dd(T) = dd(T^*) = dd(T_1^*) + 2$  which implies  $sd_{dd}(T_1) = 2$ .

By inductive hypothesis,  $T_1 \in \mathcal{F}$  and hence,  $sta(v_4) = C$ , by Observation 10. Now  $T$  can be obtained from  $T_1$  by operation  $\mathfrak{T}_1$ . Therefore  $T \in \mathcal{F}$ .

Subcase 1.2:  $\deg(v_4) \geq 3$ . Assume  $v_4$  is a support vertex or is adjacent to a support vertex. Let  $T^*$  be obtained from  $T$  by subdividing the edge  $v_1v_2$  with vertex  $x$ . Suppose that  $S$  is a  $dd(T^*)$ -set. Obviously,  $v_1, x \in S$ . In order to dominate  $v_3$  at least twice, we may assume  $v_3, v_4 \in S$ . Now  $(S \setminus \{x, v_3\}) \cup \{v_2\}$  is a double dominating set of  $T$  which contradicts  $sd_{dd}(T) = 2$ . Since  $v_1v_2 \dots v_r$  is a longest path in  $T$ , it follows that  $T_{1,v_4}$  is a star  $K_{1,t}$  ( $t = \deg_T(v_4) - 1 \geq 2$ ) whose edges

are subdivided twice. Then we may assume  $T_1$  has a path  $v_4x_1x_2x_3$  where  $x_1 \notin \{v_3, v_5\}$ . Obviously,  $dd(T) \leq dd(T_1) + 2$ . We claim that  $T$  has a  $dd(T)$ -set  $S'$  such that  $v_4 \in S'$  and  $v_3 \notin S'$ . Let  $S$  be a  $dd(T)$ -set. First let  $v_4 \in S$ . If  $v_3 \notin S$ , then we take  $S' = S$ . Let  $v_3 \in S$ . Then obviously  $x_1 \notin S$  and  $S' = (S \setminus \{v_3\}) \cup \{x_3\}$  is the desired  $dd(T)$ -set. Now let  $v_4 \notin S$ . Then  $\{x_i, v_i \mid 1 \leq i \leq 3\} \subseteq S$  and clearly  $S' = (S \setminus \{v_3\}) \cup \{v_4\}$  is a  $dd(T)$ -set. This proves our claim. Now since  $S' \setminus \{x_1, x_2\}$  is a double dominating set for  $T_1$ , it follows that  $dd(T_1) \leq dd(T) - 2$  and  $dd(T) = dd(T_1) + 2$ . Let  $T_1^*$  (respectively,  $T^*$ ) be obtained from  $T_1$  (respectively,  $T$ ) by subdividing the edge  $e \in E(T_1)$ . An argument similar to that described above shows that  $dd(T^*) = dd(T_1^*) + 2$ . Since  $sd_{dd}(T) = 2$ , we have  $dd(T_1) + 2 = dd(T) = dd(T^*) = dd(T_1^*) + 2$ . Hence,  $sd_{dd}(T_1) = 2$  and by the inductive hypotheses  $T_1 \in \mathcal{F}$ . Therefore, by Observation 10(1, 2, 3, 4),  $sta(v_4) = C$  and hence,  $T$  can be obtained from  $T_1$  by operation  $\mathfrak{T}_1$ . This implies  $T \in \mathcal{F}$ .

Case 2:  $\deg(v_3) \geq 3$ . We consider two subcases.

Subcase 2.1:  $v_3$  is adjacent to only support vertices, except possibly  $v_4$ , in  $T$ . Assume  $T$  is rooted at  $v_r$ . Then  $T_{v_3}$  is a subdivided star  $SK_{1,t-1}$  where  $t = \deg(v_3)$ . Let  $V(T_{v_3}) = \{v_1, v_2, v_3, x_1^i, x_2^i : 1 \leq i \leq t - 2\}$  and  $T_3 = T - T_{v_3}$ . Now if  $e \in E(T_3)$  is subdivided, then by Lemma 16,

$$dd(T_3) + 2(t - 1) = dd(T) = dd(T^*) = dd(T_3^*) + 2(t - 1).$$

Hence,  $dd(T_3) = dd(T_3^*)$  and  $sd_{dd}(T_3) = 2$ . Therefore by inductive hypothesis,  $T_3 \in \mathcal{F}$ . If  $v_4$  is a support vertex, then let  $T^*$  be obtained from  $T$  by subdividing the edge  $v_3v_4$  with vertex  $x$ . Let  $S$  be a  $dd(T^*)$ -set. Obviously,  $|S \cap \{v_3, x\}| \geq 1$ . Therefore  $S \setminus \{v_3, x\}$  is a double dominating set for  $T$ , a contradiction. Let  $v_4$  be adjacent to a support vertex  $y_1$  and  $y_2 \in N(y_1) \setminus \{v_4\}$ . Let  $T^*$  be obtained from  $T$  by subdividing the edge  $v_3v_4$  with vertex  $x$  and let  $S$  be a  $dd(T^*)$ -set. Then  $|S \cap \{v_3, x, v_4\}| \geq 2$ . Now  $(S \setminus \{v_3, v_4, x\}) \cup \{v_4\}$  is a double dominating set for  $T_3$ . This implies  $dd(T^*) > dd(T)$ , a contradiction. Hence,  $\deg_T(v_4) = 2$  or  $T_w$  is a subdivided star  $SK_{1,\deg(w)-1}$  for each  $w \in N(v_4) \setminus \{v_3, v_5\}$ . If  $\deg_T(v_4) = 2$ , then  $\deg_{T_3}(v_4) = 1$  and since  $T_3 \in \mathcal{F}$ , we see that  $sta(v_4) = C$ , by Observation 10. If  $\deg(v_4) \geq 3$  and  $w \in N(v_4) \setminus \{v_3, v_5\}$ , then by Observation 10 we see that  $sta(w) = A$  and  $N(w) \setminus \{v_4\} \subseteq B(T)$ . Therefore,  $sta(v_4) = C$ . Now  $T$  can be obtained from  $T_3$  by applying operation  $\mathfrak{T}_1$  once and operation  $\mathfrak{T}_2$ ,  $(\deg(v_3) - 2)$  times. Hence,  $T \in \mathcal{F}$ .

Subcase 2.2:  $v_3$  is adjacent to a leaf  $x$ . Let  $T_1 = T \setminus \{v_1, v_2\}$ . For any  $dd(T_1)$ -set  $S$ ,  $S \cup \{v_1, v_2\}$  is a double dominating set for  $T$ . So  $dd(T) \leq dd(T_1) + 2$ . Let  $S$  be a  $dd(T)$ -set. Obviously,  $\{v_1, v_2, v_3, x\} \subseteq S$ . Now  $S \setminus \{v_1, v_2\}$  is a double dominating set for  $T_1$ . Therefore,  $dd(T) = dd(T_1) + 2$ .

First let  $\deg(v_3) \geq 4$ . Then  $v_3$  is adjacent to a support vertex, say  $y$ , by Proposition 3. Let  $e \in E(T_1)$  and  $T_1^*$  (respectively,  $T^*$ ) be obtained from  $T_1$  (respectively,  $T$ ) by subdividing the edge  $e$ . It is easy to see that  $dd(T^*) = dd(T_1^*) + 2$ . Since  $sd_{dd}(T) = 2$ , we have  $dd(T_1^*) + 2 = dd(T^*) = dd(T) = dd(T_1) + 2$  and hence,  $sd_{dd}(T_1) = 2$ . By inductive hypotheses  $T_1 \in \mathcal{F}$  and so  $sta(v_3) = B$  by Observation 10(2). Now  $T$  can be obtained from  $T_1$  by operation  $\mathfrak{T}_2$  and so  $T \in \mathcal{F}$ .

Now let  $\deg(v_3) = 3$ . First assume  $\deg(v_4) = 2$ . Let  $e \in E(T_1)$  and  $T_1^*$  (respectively,  $T^*$ ) be obtained from  $T_1$  (respectively,  $T$ ) by adding a new vertex  $z$  which subdivides the edge  $e$ . If  $e \neq v_3x$ , then obviously  $dd(T^*) = dd(T_1^*) + 2$ . Now let  $e = v_3x$  and let  $S$  be a  $dd(T^*)$ -set. Then  $v_3 \notin S$ , otherwise  $S \setminus \{z\}$  is a double dominating set for  $T$  which leads to  $dd(T) < dd(T^*)$ , a contradiction. Now since  $\deg(v_4) = 2$ , we must have  $v_4 \in S$ . Hence,  $S \setminus \{v_1, v_2\}$  is a double dominating set for  $T_1^*$ . Thus,  $dd(T^*) = dd(T_1^*) + 2$  and we obtain  $sd_{dd}(T_1) = 2$ . As above this leads to  $T \in \mathcal{F}$ . Now let  $\deg(v_4) \geq 3$ . First assume  $v_4$  is a support vertex. Let  $e \in E(T_1)$  be subdivided. Since for any  $dd(T^*)$ -set  $S$ ,  $|S \cap \{v_3, v_4\}| \geq 1$ , it is easy to see that  $dd(T^*) = dd(T_1^*) + 2$  and hence,  $sd_{dd}(T_1) = 2$ . This leads to  $T \in \mathcal{F}$ . Now we assume that  $v_4$  is not a support vertex. Let  $y \in N(v_4) \setminus \{v_5, v_3\}$ . If the height of  $T_y$  is 1, then  $\deg(y) = 2$  by Proposition 3. If the height of  $T_y$  is 2, then by repeating above argument we may assume  $T_y \simeq T_{v_3}$ .

First assume  $\deg(v_4) \geq 4$ . Let  $T_2 = T - T_{v_3}$ ,  $e \in E(T_2)$  and  $T_1^*$  (respectively,  $T^*$ ) be obtained from  $T_1$  (respectively,  $T$ ) by subdividing the edge  $e$ . It is easy to see that  $dd(T) = dd(T_2) + 4$  and  $dd(T^*) = dd(T_2^*) + 4$  which implies  $sd_{dd}(T_2) = 2$ . By inductive hypotheses  $T_2 \in \mathcal{F}$ . Since every neighbor of  $v_4$  in  $T_2$  (except possibly  $v_5$ ) is a support vertex,  $sta(v_4) = A$  or  $B$  by Observation 10(4). Now  $T$  can be obtained from  $T_2$  by applying operation  $\mathfrak{T}_2$  twice and hence,  $T \in \mathcal{F}$ .

Now assume  $\deg(v_4) = 3$  and  $y \in N(v_4) \setminus \{v_3, v_5\}$ . Recall that the height of  $T_y$  is 1 or 2. Suppose that  $T_y \simeq T_{v_3}$  ( $T_y \simeq K_2$ ). That is  $T_y = zy_1y_2$  (respectively,  $T_y = yz$ ). If  $v_5$  is either a support vertex or adjacent to a support vertex and if  $T^*$  is obtained from  $T$  by subdividing the edge  $v_4v_5$ , then it is easy to see that  $dd(T^*) > dd(T)$ , a contradiction. Therefore,  $v_5$  is not a support vertex or adjacent to a support vertex. Let there exists  $w \in N(v_5) \setminus \{v_4, v_6\}$  such that the height of  $T_w$  is 2. Then  $T_w$  is a subdivided star  $K_{1,\deg(w)-1}$ , by Proposition 3. Let  $T_3 = T - T_w$ . Since  $sd_{dd}(T) = 2$ , it follows that  $sd_{dd}(T_3) = 2$ , by Lemma 16. By inductive hypotheses  $T_3 \in \mathcal{F}$ . It is easy to see that  $sta(v_5) = C$  and hence,

$T$  can be obtained from  $T_3$  by applying operation  $\mathfrak{T}_1$  once and operation  $\mathfrak{T}_2$ ,  $\deg(w) - 2$  times. Thus,  $T \in \mathcal{F}$ . Finally, let the height of subtree  $T_w$  be four for every  $w \in N(v_5) \setminus \{v_4, v_6\}$ . Suppose that  $T_4 = T - T_{v_4}$ . For any  $dd(T_4)$ -set  $S$ ,  $S \cup (V(T_{v_4}) - \{v_4\})$  is a double dominating set for  $T$ . Therefore if  $T_y \simeq T_{v_3}$ , then  $dd(T) \leq dd(T_4) + 8$ , and if  $T_y \simeq K_2$ , then  $dd(T) \leq dd(T_4) + 6$ , where  $y \in N(v_4) \setminus \{v_3, v_5\}$ . Now let  $D$  be a  $dd(T)$ -set. If  $v_4 \notin D$ , then  $D' = D \setminus V(T_{v_4})$  is a double dominating set for  $T_4$ , and if  $v_4 \in D$ , then  $D' = (D \setminus V(T_{v_4})) \cup \{u\}$  is a double dominating set for  $T_4$ , where  $u \in N(v_5) \setminus D$ . Hence,  $dd(T) = dd(T_4) + 8$  if  $T_y \simeq T_{v_3}$  and  $dd(T) = dd(T_4) + 6$  if  $T_y \simeq K_2$ . Let  $e \in E(T_4)$  and let  $T_4^*$  (respectively,  $T^*$ ) be obtained from  $T_4$  (respectively,  $T$ ) by subdividing the edge  $e$ . In a similar fashion, we can show that  $dd(T^*) = dd(T_4^*) + 8$  (respectively,  $dd(T^*) = dd(T_4^*) + 6$ ). Thus,  $sd_{dd}(T_4) = 2$  and so  $T_4 \in \mathcal{F}$  by inductive hypotheses. By Observation 10, we see that  $sta(v_5) = C$ . Now  $T$  can be obtained from  $T_4$  by applying operation  $\mathfrak{T}_1$  once and operation  $\mathfrak{T}_2$ , three times if  $T_y \simeq T_{v_3}$  (twice if  $T_y \simeq K_2$ ). Thus,  $T \in \mathcal{F}$ . This completes the proof.  $\square$

We conclude this paper with the following problem.

**Problem.** Prove or disprove: let  $G$  be a connected graph with no isolated vertices. Then  $1 \leq sd_{dd}(G) \leq 2$ .

## References

- [1] H. Aram, O. Favaron, S.M. Sheikholeslami, Trees with domination subdivision number three, submitted for publication.
- [2] A. Bhattacharya, G.R. Vijayakumar, Effect of edge-subdivision on vertex-domination in a graph, *Discuss. Math. Graph Theory* 22 (2002) 335–347.
- [3] S. Brandt, Subtrees and subforests of graphs, *J. Combin. Theory Ser. B* 61 (1994) 63–70.
- [4] O. Favaron, T.W. Haynes, S.T. Hedetniemi, Domination subdivision numbers in graphs, *Utilitas Math.* 66 (2004) 195–209.
- [5] F. Harary, T.W. Haynes, Double domination in graphs, *Ars Combin.* 55 (2000) 201–213.
- [6] T.W. Haynes, S.M. Hedetniemi, S.T. Hedetniemi, J. Knisely, L.C. van der Merwe, Domination subdivision numbers, *Discuss. Math. Graph Theory* 21 (2001) 239–253.
- [7] T.W. Haynes, S.T. Hedetniemi, L.C. van der Merwe, Total domination subdivision numbers, *J. Combin. Math. Combin. Comput.* 44 (2003) 115–128.
- [8] T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York, 1998.
- [9] T.W. Haynes, M.A. Henning, L.S. Hopkins, Total domination subdivision numbers of trees, *Discrete Math.* 286 (2004) 195–202.
- [10] T.W. Haynes, M.A. Henning, L.S. Hopkins, Total domination subdivision numbers of graphs, *Discuss. Math. Graph Theory* 24 (2004) 457–467.
- [11] H. Karami, A. Khodkar, R. Khoeilar, S.M. Sheikholeslami, Trees whose total domination subdivision numbers is one, *Bull. Inst. Combin. Appl.* to appear.
- [12] D.B. West, *Introduction to Graph Theory*, Prentice-Hall, Englewood Cliffs, 2000.