The Structure of Bull-Free Perfect Graphs

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Abstract: The *bull* is a graph consisting of a triangle and two vertex-disjoint pendant edges. A graph is called *bull-free* if no induced subgraph of it is a bull. A graph G is *perfect* if for every induced subgraph H of G, the chromatic number of H equals the size of the largest complete subgraph of H. This article describes the structure of all bull-free perfect graphs.
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1. INTRODUCTION

Unless stated otherwise, all graphs in this article are simple and finite. Given a graph G, we denote by V_G the vertex set of G, and by E_G the edge set of G; we sometimes write $G = (V_G, E_G)$. A *clique* in G is a set of pairwise adjacent vertices of G, and a *stable set* in G is a set of pairwise nonadjacent vertices of G. The clique number of G, written $\omega(G)$, is the maximum number of vertices in a clique in G. A *coloring* of G is a partition of V_G into stable sets. The chromatic number of a graph G, written $\chi(G)$, is the minimum

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number of stable sets needed to partition V_G. A graph G is said to be *perfect* if for every induced subgraph H of G, $\chi(H) = \omega(H)$.

Given a graph G, the complement of G, written $\overline{G}$, is the graph on the vertex set V_G such that for all distinct $u, v \in V_G$, u is adjacent to v in $\overline{G}$ if and only if u is nonadjacent to v in G. A *hole* in G is an induced cycle of length at least four in G. An *antihole* in G is an induced subgraph of G whose complement is a hole in $\overline{G}$. An *odd hole* (respectively: *odd antihole*) is a hole (respectively: anti-hole) with an odd number of vertices. Berge conjectured that a graph is perfect if and only if it contains neither odd holes nor odd antiholes [1]. This conjecture is known as the strong perfect graph conjecture; it was proven a few years ago [7], and it is now known as the strong perfect graph theorem. Today, graphs that contain no odd holes and no odd antiholes are called *Berge*, and so the strong perfect graph theorem can be restated as saying that a graph is perfect if and only if it is Berge. Other relevant results have been the polynomial time recognition algorithm for Berge (and therefore, by the strong perfect graph theorem, perfect) graphs [6], and a polynomial time coloring algorithm for perfect graphs [12]. As the latter algorithm is based on the ellipsoid method, finding a combinatorial polynomial time coloring algorithm for perfect graphs remains an open problem.

The *bull* is a graph with vertex set $\{x_1, x_2, x_3, y_1, y_2\}$ and edge set $\{x_1x_2, x_2x_3, x_3x_1, x_1y_1, x_2y_2\}$. A graph G is said to be *bull-free* if no induced subgraph of G is isomorphic to the bull.

Bull-free graphs were originally studied in the context of perfect graphs. Many years before the strong perfect graph conjecture was proven [7], it was shown that bull-free Berge graphs were perfect [8]. Similarly, a polynomial time recognition algorithm for bull-free perfect graphs [15] was obtained long before the recognition algorithm for perfect graphs [6]. Furthermore, the class of bull-free perfect graphs is one of the subclasses of the class of perfect graphs for which a combinatorial polynomial time coloring algorithm has been constructed [10] (in fact, [10] contains a weighted coloring algorithm for bull-free perfect graphs; see also [11] and [13]). Recently, a structure theorem for bull-free graphs was obtained [2–5]; in the present article, we use the structure theorem from [2–5] to derive a structure theorem for bull-free Berge graphs; by the strong perfect graph theorem [7], this is in fact a structure theorem for bull-free perfect graphs. While the structure of bull-free perfect graphs has received some attention in the past (see [8] and [11]), all previous results were "decomposition theorems": they state that every bull-free perfect graph either belongs to some well-understood "basic" class or it can be "decomposed" into smaller bull-free perfect graphs in a certain useful way. However, these decompositions cannot be turned into operations that build larger graphs from smaller ones, while preserving the property of being bull-free and perfect. In this sense, the structure theorem from the present article is stronger: it states that every bull-free perfect graph either belongs to a basic class, or it can be built from smaller bull-free perfect graph by an operation that preserves the property of being bull-free and perfect.

We remark that elsewhere [14], the results of the present article are used to obtain a combinatorial polynomial time weighted coloring algorithm for bull-free perfect graphs whose complexity is lower than that of the algorithm in [10]. One might ask whether these results could also be used as a basis for a polynomial time recognition algorithm for bull-free perfect graphs. While it is possible that such an algorithm could be found, it seems unlikely that it would be faster than the recognition algorithm for bull-free perfect graphs from [15]; this is because the running time of the algorithm from [15] is only $O(n^5)$ (where n is the number of vertices of the input graph), and since the bull contains five vertices, the obvious way to test whether a graph is bull-free already takes $O(n^5)$ time.

Following [2–5], we consider objects called "trigraphs" (defined in Section 2), which are a generalization of graphs. While in a graph, two distinct vertices can be either adjacent or nonadjacent, in a trigraph, there are three options: a pair of distinct vertices can be adjacent, antiadjacent, or semiadjacent; semiadjacent pairs can conveniently be thought of as having undecided adjacency. While we do not define such a thing as a "perfect trigraph" (because we do not have a natural way to define a trigraph coloring), there is a natural way to define a "bull-free trigraph" and a "Berge trigraph," and we do this in Section 2. Our main theorem (3.4) gives the structure of all bull-free Berge trigraphs. Since graphs are simply trigraphs with no semiadjacent pairs, this theorem is implicitly a structure theorem for bull-free Berge graphs, and therefore (by the strong perfect graph theorem) it is a structure theorem for bull-free perfect graphs.

The rest of the article is organized as follows. In Section 3, we define "elementary trigraphs," and we use a result from [2] to reduce our problem to finding the structure of all elementary bull-free Berge trigraphs. We then cite the structure theorem for elementary bull-free trigraphs from [5]; this theorem states that every bull-free trigraph G is either obtained from smaller bull-free trigraphs by "substitution" (this operation is defined in Section 2), or G or its complement is an "elementary expansion" (this is defined later, in Section 4) of a trigraph in one of two basic classes (classes $\mathcal{T}_1$ and $\mathcal{T}_2$). We complete Section 3 by stating our main theorem (3.4), the structure theorem for all bull-free Berge trigraphs; however, we do not prove this theorem in Section 3 (we only do this in Section 7), and we also postpone defining certain terms used in the theorem. In Section 4, we study "elementary expansions." Informally, an elementary expansion of a trigraph H is the trigraph obtained by expanding some semiadjacent pairs of H to "homogeneous pairs" of a certain kind (general homogeneous pairs are defined in Section 2, and the two kinds that we need for elementary expansions are defined in Section 4). We show that if G is an elementary expansion of a trigraph H, then G is Berge if and only if H is. In Section 5, we give the definition of the class $\mathcal{T}_1$ from [5] and derive the class $\mathcal{T}_1^*$ of all Berge trigraphs in $\mathcal{T}_1$. In Section 6, we define the class $\mathcal{T}_2$ from [5], and prove that every trigraph in $\mathcal{T}_2$ is Berge. Finally, in Section 7, we prove our main theorem.

2. TRIGRAPHS

A trigraph G is an ordered pair (V_G, θ_G) such that V_G is a nonempty finite set, called the *vertex set* of G, and $\theta_G : V_G^2 \to \{-1, 0, 1\}$ is a map, called the *adjacency function* of G, satisfying the following:

- for all $v \in V_G$, $\theta_G(v, v) = 0$;
- for all $u, v \in V_G$, $\theta_G(u, v) = \theta(v, u)$;
- for all $u \in V_G$, there exists at most one vertex $v \in V_G \setminus \{u\}$ such that $\theta_G(u, v) = 0$.

Members of V_G are called *vertices* of G. Let $u, v \in V_G$ be distinct. We say that uv is a *strongly adjacent pair*, or that u and v are *strongly adjacent*, or that u is *strongly adjacent* to v, or that u is a *strong neighbor* of v, provided that $\theta_G(u, v) = 1$. We say that uv is a *strongly antiadjacent pair*, or that u and v are *strongly antiadjacent*, or that u is *strongly antiadjacent* to v, or that u is a *strong antineighbor* of v, provided that

$\theta_G(u, v) = -1$. We say that uv is a *semiadjacent pair*, or that u and v are *semiadjacent*, or that u is *semiadjacent* to v, provided that $\theta_G(u, v) = 0$. (Note that we do not say that a vertex $w \in V_G$ is semiadjacent to itself even though $\theta_G(w, w) = 0$.) If uv is a strongly adjacent pair or a semiadjacent pair, then we say that uv is an *adjacent pair*, or that u and v are *adjacent*, or that u is *adjacent* to v, or that u is a *neighbor* of v. If uv is a strongly antiadjacent pair or a semiadjacent pair, then we say that uv is an *antiadjacent pair*, or that u and v are *antiadjacent*, or that u is *antiadjacent* to v, or that u is an *antineighbor* of v. Thus, if uv is a semiadjacent pair, then uv is simultaneously an adjacent pair and an antiadjacent pair. The *endpoints* of the pair uv (regardless of adjacency) are u and v. Given distinct vertices u and v of G, we do not distinguish between pairs uv and vu. However, we will sometimes need to maintain the asymmetry between the endpoints of a semiadjacent pair uv, and in those cases, we will use the ordered pair notation and write (u, v) or (v, u), as appropriate, rather than uv.

Note that every (nonempty, finite, and simple) graph can be thought of as a trigraph in a natural way: graphs are simply trigraphs with no semiadjacent pairs.

The *complement* of a trigraph G is the trigraph $\overline{G}$ such that $V_{\overline{G}} = V_G$ and $\theta_{\overline{G}} = -\theta_G$; we say that a class $\mathcal{C}$ of trigraphs is *self-complementary* provided that for every trigraph $G \in \mathcal{C}$, we have that $\overline{G} \in \mathcal{C}$. Given a trigraph G and a nonempty set $X \subseteq V_G$, $G[X]$ is the trigraph with vertex set X and adjacency function $\theta_G \restriction X^2$ (the restriction of θ_G to X^2); we call $G[X]$ the *subtrigraph of* G induced by X. Isomorphism between trigraphs is defined in the natural way; if trigraphs G_1 and G_2 are isomorphic, then we write $G_1 \cong G_2$. Given trigraphs G and H, we say that H is an *induced subtrigraph* of G (or that G *contains* H *as an induced subtrigraph*) if there exists some $X \subseteq V_G$ such that $H = G[X]$. (However, when convenient, we relax this condition and say that H is an induced subtrigraph of G, or that G contains H as an induced subtrigraph, if there exists some $X \subseteq V_G$ such that $H \cong G[X]$.) Given a trigraph G and a set $X \subseteq V_G$, we denote by $G \setminus X$ the trigraph $G[V_G \setminus X]$; if $X = \{v\}$, then we sometimes write $G \setminus v$ instead of $G \setminus \{v\}$.

Given a trigraph G, a vertex $a \in V_G$, and a set $B \subseteq V_G \setminus \{a\}$, we say that a is *strongly complete* (respectively: *strongly anticomplete*, *complete*, *anticomplete*) to B provided that a is strongly adjacent (respectively: strongly antiadjacent, adjacent, antiadjacent) to every vertex in B; we say that a is *mixed* on B provided that a is neither strongly complete nor strongly anticomplete to B. Given disjoint $A, B \subseteq V_G$, we say that A is *strongly complete* (respectively: *strongly anticomplete*, *complete*, *anticomplete*) to B provided that for every $a \in A$, a is strongly complete (respectively: strongly anticomplete, complete, anticomplete) to B. Given $X \subseteq V_G$, we say that X is a *(strong) clique* in G provided that the vertices of X are all pairwise (strongly) adjacent, and we say that X is a *(strongly) stable set* in G provided that the vertices of X are all pairwise (strongly) antiadjacent. A *(strong) triangle* is a (strong) clique consisting of three vertices, and a *(strong) triad* is a (strongly) stable set consisting of three vertices.

A graph $\hat{G}$ is said to be a *realization* of a trigraph $G = (V_G, \theta_G)$ provided that the vertex set of $\hat{G}$ is V_G, and for all distinct $u, v \in V_G$, the following hold:

- if u is adjacent to v in $\hat{G}$, then u is adjacent to v in G;
- if u is nonadjacent to v in $\hat{G}$, then u is antiadjacent to v in G.

Thus, a realization of a trigraph is obtained by turning all strongly adjacent pairs into edges, all strongly antiadjacent pairs into nonedges, and semiadjacent pairs (independently of one another) into edges or nonedges.

A *bull* is a trigraph with vertex set $\{x_1, x_2, x_3, y_1, y_2\}$ such that $\{x_1, x_2, x_3\}$ is a triangle, y_1 is adjacent to x_1 and anticomplete to $\{x_2, x_3, y_2\}$, and y_2 is adjacent to x_2 and anticomplete to $\{x_1, x_3\}$. We say that a trigraph G is *bull-free* provided that no induced subtrigraph of G is a bull.

We say that an induced subtrigraph P of a trigraph G is a *path* in G provided that the vertices of P can be ordered as $p_0, p_1, \dots, p_k$ (for some integer $k \geq 0$) so that for all $i, j \in \{0, \dots, k\}$, if $|i-j| = 1$ then p_ip_j is an adjacent pair, and if $|i-j| > 1$ then p_ip_j is an antiadjacent pair; we often denote such a path P by $p_0 - p_1 - \cdots - p_k$, and we say that P is a path *from* p_0 to p_k, or that P is a path *between* p_0 and p_k. The *length* of a path or antipath P is $|V_P| - 1$, where (consistently with our notation) V_P denotes the vertex set of P. If P is a path of length k, then we also say that P is a *k-edge path*. An *odd path* is a path of odd length.

We say that an induced subtrigraph H of a trigraph G is a *hole* (respectively: *antihole*) in G provided that the vertices of H can be ordered as $h_1, \dots, h_k$ (for some integer $k \geq 4$) so that for all $i, j \in \{1, \dots, k\}$, if $|i-j| = 1$ or $|i-j| = k-1$ then h_ih_j is an adjacent (respectively: antiadjacent) pair, and if $1 < |i-j| < k-1$ then h_ih_j is an antiadjacent (respectively: adjacent) pair. We often denote such a hole or antihole by $h_1 - h_2 - \cdots - h_k - h_0$. (We observe that H is a hole in G if and only if $\overline{H}$ is an antihole in $\overline{G}$.) The *length* of a hole or antihole H is $|V_H|$. If H is a hole (respectively: antihole) of length k, then we also say that H is a *k-hole* (respectively: *k-antihole*). We say that H is an *odd hole* (respectively: *odd antihole*) provided that H is a hole (respectively: antihole) of length k for some odd integer $k \geq 5$. We say that a trigraph G is *odd hole-free* (respectively: *odd antihole-free*) provided that no induced subtrigraph of G is an odd hole (respectively: odd antihole). A trigraph G is *Berge* provided that G is both odd hole-free and odd antihole-free.

We say that a trigraph G is *bipartite* provided that V_G can be partitioned into (possibly empty) strongly stable sets A and B; under such circumstances, we say that G is bipartite with *bipartition* (A, B), and that (A, B) is a bipartition of the bipartite trigraph G. We say that a trigraph G is *complement-bipartite* provided that $\overline{G}$ is bipartite. We say that G is complement-bipartite with *bipartition* (A, B), or that (A, B) is a bipartition of the complement-bipartite trigraph G, provided that $\overline{G}$ is bipartite with bipartition (A, B).

We observe that a trigraph G is bull-free (respectively: odd hole-free, odd antihole-free, Berge, bipartite, complement-bipartite) provided that every realization of G is bull-free (respectively: odd hole-free, odd antihole-free, Berge, bipartite, complement-bipartite).

We note that bipartite and complement-bipartite trigraphs are Berge. This follows from the fact that every realization of a bipartite (respectively: complement-bipartite) trigraph is a bipartite (respectively: complement-bipartite) graph, and it is a well-known (and easy to check) fact that bipartite and complement-bipartite graphs are Berge.

We now give an easy result that will be used throughout the article.

2.1. *Let G be a trigraph. Then G is bull-free if and only if $\overline{G}$ is bull-free, and G is Berge if and only if $\overline{G}$ is Berge.*

Proof. The first claim follows from the fact that the complement of a bull is again a bull. The second claim is immediate from the definition. ■

Let G be a trigraph, and let $\hat{G}$ be the realization of G that satisfies the property that for all distinct $u, v \in V_G$, u and v are adjacent in $\hat{G}$ if and only if $\theta_G(u, v) \geq 0$. (Thus, $\hat{G}$ is the realization of G obtained by turning all semiadjacent pairs of G into edges.) We then say that G is *connected* provided that $\hat{G}$ is connected. Let $X \subseteq V_G$. We say that $G[X]$ is a *component* of G provided that $\hat{G}[X]$ is a *component* of $\hat{G}$ (i.e., provided that $\hat{G}[X]$ is a maximal connected induced subgraph of $\hat{G}$).

Let G be a trigraph, and let X be a proper, nonempty subset of V_G. We say that X is a *homogeneous set* in G provided that for every $v \in V_G \setminus X$, v is either strongly complete to X or strongly anticomplete to X. We say that X is a *proper* homogeneous set in G provided that X is a homogeneous set in G and that $|X| \geq 2$. We say that G admits a *homogeneous set decomposition* provided that G contains a proper homogeneous set. We observe that if X is a homogeneous set in G, and uv is a semiadjacent pair in G, then either $u, v \in X$ or $u, v \in V_G \setminus X$.

Let G, G_1, and G_2 be trigraphs, and let $v \in V_{G_1}$. Assume that V_{G_1} and V_{G_2} are disjoint, and that v is not an endpoint of any semiadjacent pair in G_1. We then say that G is obtained by *substituting* G_2 for v in G_1 provided that all of the following hold:

- $V_G = (V_{G_1} \cup V_{G_2}) \setminus \{v\}$;
- $G[V_{G_1} \setminus \{v\}] = G_1 \setminus v$;
- $G[V_{G_2}] = G_2$;
- for all $v_1 \in V_{G_1} \setminus \{v\}$ and $v_2 \in V_{G_2}$, v_1v_2 is a strongly adjacent pair in G if v_1v is a strongly adjacent pair in G_1, and v_1v_2 is a strongly antiadjacent pair in G if v_1v is a strongly antiadjacent pair in G_1.

We say that a trigraph G is obtained by *substitution from smaller trigraphs* provided that there exist some trigraphs G_1 and G_2 with disjoint vertex sets satisfying $|V_{G_1}| < |V_G|$ and $|V_{G_2}| < |V_G|$ (or equivalently: $|V_{G_1}| \geq 2$ and $|V_{G_2}| \geq 2$) and some $v \in V_{G_1}$ that is not an endpoint of any semiadjacent pair in G_1, such that G is obtained by substituting G_2 for v in G_1. We observe that a trigraph G admits a homogeneous set decomposition if and only if G is obtained from smaller trigraphs by substitution. We will use the following result several times in this article.

2.2. *Let G, G_1, G_2 be trigraphs and let $v \in V_{G_1}$. Assume that V_{G_1} and V_{G_2} are disjoint and that v is not an endpoint of any semiadjacent pair in G_1. Assume that G is obtained by substituting G_2 for v in G_1. Then all of the following hold:*

- *G is bull-free if and only if G_1 and G_2 are both bull-free;*
- *G is odd hole-free if and only if both G_1 and G_2 are odd hole-free;*
- *G is odd antihole-free if and only if both G_1 and G_2 are odd antihole-free;*
- *G is Berge if and only if G_1 and G_2 are both Berge.*

Proof. This is an easy consequence of the fact that bulls, holes of length at least five, and antiholes of length at least five do not admit a homogeneous set decomposition. ■

Next, let G be a trigraph, and let A and B be nonempty, disjoint subsets of V_G. We say that (A, B) is a *homogeneous pair* in G provided that A is a homogeneous set in $G \setminus B$, and B is a homogeneous set in $G \setminus A$. We observe that if (A, B) is a homogeneous pair in G, and uv is a semiadjacent pair in G, then either $u, v \in A \cup B$ or $u, v \in V_G \setminus (A \cup B)$. If (A, B) is a homogeneous pair in G, then the *partition of G with respect to* (or *associated with*) the homogeneous pair (A, B) is the 6-tuple (A, B, C, D, E, F), where C is the set of all vertices in $V_G \setminus (A \cup B)$ that are strongly complete to A and strongly anticomplete to B; D is the set of all vertices in $V_G \setminus (A \cup B)$ that are strongly complete to B and strongly anticomplete to A; E is the set of all vertices in $V_G \setminus (A \cup B)$ that are strongly complete to $A \cup B$; and F is the set of all vertices in $V_G \setminus (A \cup B)$ that are strongly anticomplete to $A \cup B$.

We say that a homogeneous pair (A, B) in a trigraph G is *good* provided that the following three conditions hold:

- neither $G[A]$ nor $G[B]$ contains a three-edge path;
- there does not exist a path $v_1 - v_2 - v_3 - v_4$ in G such that $v_1, v_4 \in A$ and v_2, $v_3 \in B$;
- there does not exist a path $v_1 - v_2 - v_3 - v_4$ in G such that $v_1, v_4 \in B$ and v_2, $v_3 \in A$.

We observe that if (A, B) is a good homogeneous pair in a trigraph G, then (A, B) is a good homogeneous pair in $\overline{G}$ as well; this follows from the fact that the complement of a three-edge path is again a three-edge path. Good homogeneous pairs will appear in Sections 4 and 6 below. We end this section with a useful result concerning good homogeneous pairs.

2.3. *Let G be a trigraph, let (A, B) be a good homogeneous pair in G, and let W be the vertex set of an odd hole or an odd antihole in G. Then $|W \cap A| \leq 1$ and $|W \cap B| \leq 1$.*

Proof. First, by passing to $\overline{G}$ if necessary, we may assume that W is the vertex set of an odd hole in G. Next, let $\hat{G}$ be a realization of G in which W is the vertex set of an odd hole. Finally, let (A, B, C, D, E, F) be the partition of G with respect to (A, B). We begin by proving that $W \not\subseteq A \cup B$. Suppose otherwise. Since the number of edges in $\hat{G}[W]$ with one endpoint in A and the other one in B is even, exactly one of $\hat{G}[W \cap A]$ and $\hat{G}[W \cap B]$ contains an odd number of edges; by symmetry, we may assume that $\hat{G}[B]$ contains an odd number of edges. Since $\hat{G}[W \cap B]$ contains no induced three-edge path, and since $\hat{G}[W]$ is a chordless cycle of length at least five, we know that $\hat{G}[W \cap B]$ contains an edge b_1b_2 that meets no other edges in $\hat{G}[W \cap B]$. Since $\hat{G}[W]$ is a chordless cycle of length at least five, there exist some $a_1, a_2 \in W$ such that $a_1 - b_1 - b_2 - a_2$ is an induced three-edge path in $\hat{G}[W]$ (and therefore in $G[W]$ as well). Since the edge b_1b_2 meets no other edges in $\hat{G}[W \cap B]$, we know that $a_1, a_2 \in A$. But then the path $a_1 - b_1 - b_2 - a_2$ contradicts the fact that (A, B) is good.

We next show that $|W \cap A| \leq 2$ and $|W \cap B| \leq 2$. Suppose otherwise. By symmetry, we may assume that $|W \cap A| \geq 3$. Then $W \cap (C \cup E) = \emptyset$, for otherwise, some vertex in $\hat{G}[W]$ would be of degree at least three. Since $W \not\subseteq A \cup B$, W intersects $D \cup F$; and since $\hat{G}[W]$ is connected, $W \cap D \neq \emptyset$. Now, fix some $a \in W \cap A$ and $d \in W \cap D$. Note that there are two paths in $\hat{G}[W]$ between a and d that meet only at their endpoints; both of these paths pass through B, and so $|W \cap B| \geq 2$. Fix distinct $b_1, b_2 \in W \cap B$. Since B is complete to D in $\hat{G}$, and since $\hat{G}[W]$ is a chordless cycle of length at least five, it follows that $W \cap B = \{b_1, b_2\}$ and $W \cap D = \{d\}$. It then easily follows that $W \setminus \{d\} \subseteq A \cup B$. Then $\hat{G}[W \cap A]$ is an odd path, and so since $|W \cap A| \geq 3$, we get that $\hat{G}[A]$ (and therefore $G[A]$) contains an induced three-edge path, contrary to the fact that (A, B) is good. Thus, $|W \cap A| \leq 2$ and $|W \cap B| \leq 2$.

Finally, suppose that $|W \cap A| = 2$; set $W \cap A = \{a_1, a_2\}$. Since $\hat{G}[W]$ is a chordless cycle of length at least five, there exist some $b_1, b_2 \in W \setminus \{a_1, a_2\}$ such that a_1b_1, a_2b_2 are edges, and a_1b_2, a_2b_1 are nonedges in $\hat{G}$. Since b_1 and b_2 are both mixed on A, it follows that $b_1, b_2 \in B$; since $|W \cap B| \leq 2$, this means that $W \cap B = \{b_1, b_2\}$. Since

(A, B) is good, and since $\hat{G}[W]$ contains no cycles of length four, we know that both a_1a_2 and b_1b_2 are nonedges. Note that $W \cap E = \emptyset$, for otherwise, some vertex in W would be of degree at least four in $\hat{G}[W]$. Thus, all neighbors of a_1 in $\hat{G}[W]$ lie in $C \cup \{b_1\}$; since a_1 has at least two neighbors in W, this means that $W \cap C \neq \emptyset$. Similarly, $W \cap D \neq \emptyset$. But if $c \in W \cap C$ and $d \in W \cap D$, then $c - a_1 - b_1 - d - b_2 - a_2 - c$ is a (not necessarily induced) cycle of length six in $\hat{G}[W]$, which is impossible. Thus, $|W \cap A| \leq 1$. In an analogous way, we get that $|W \cap B| \leq 1$. This completes the argument. ■

3. STRUCTURE THEOREM FOR BULL-FREE BERGE TRIGRAPHS

Given a trigraph G, a set $X \subseteq V_G$, and a vertex $p \in V_G \setminus X$, we say that p is a *center* for X (or for $G[X]$) provided that p is adjacent to each vertex in X, and we say that p is an *anticenter* for X (or for $G[X]$) provided that p is antiadjacent to each vertex in X. Following [2], we call a bull-free trigraph G *elementary* provided that it contains no three-edge path P such that some vertex of G is a center for P, and some vertex of G is an anticenter for P. A bull-free trigraph that is not elementary is said to be *nonelementary*. We now state a decomposition theorem from [2] (this is 3.3 from [2]; we remark that not all terms from the statement of this theorem have been defined in the present article).

3.1. *Let G be a nonelementary bull-free trigraph. Then at least one of the following holds:*

- *G or $\overline{G}$ belongs to $\mathcal{T}_0$;*
- *G or $\overline{G}$ contains a homogeneous pair of type zero;*
- *G admits a homogeneous set decomposition.*

We omit the definitions of $\mathcal{T}_0$ and of a homogeneous pair of type zero, and instead refer the reader to [2]. What we need here is the fact (easy to check) that every trigraph in $\mathcal{T}_0$ contains a hole of length five, as does every trigraph that contains a homogeneous pair of type zero. Now 3.1 (i.e., 3.3 from [2]) implies that every nonelementary bull-free trigraph that does not contain a hole of length five (and in particular, every nonelementary bull-free Berge trigraph) admits a homogeneous set decomposition; we state this result below for future reference.

3.2. *Every nonelementary bull-free trigrpaph that does not contain a hole of length five admits a homogeneous set decomposition. In particular, every nonelementary bull-free Berge trigraph admits a homogeneous set decomposition.*

While the proof of 3.3 from [2] (stated as 3.1 above) is relatively involved, if we restrict our attention to nonelementary bull-free trigraphs G that do not contain a hole of length five, only a couple of pages are needed to prove that G admits a homogeneous set decomposition (we refer the reader to the proof of 5.2 from [2]). We also remark that for the case of graphs (rather than trigraphs), a result analogous to 3.2 was originally proven in [11].

Recall that a trigraph G admits a homogeneous set decomposition if and only if it can be obtained from smaller trigraphs by substitution. Since the class of bull-free Berge trigraphs is closed under substitution (by 2.2), we need only consider bull-free Berge trigraphs that do not admit a homogeneous set decomposition, and by 3.2, all such trigraphs are elementary. Thus, the rest of the article deals with bull-free Berge trigraphs that are elementary.

We now state the structure theorem for elementary bull-free trigraphs. (We note that some terms used in the statement of this theorem have not yet been defined.) The following is an immediate consequence of 6.1 and 5.5 from [5].

3.3. *Let G be an elementary bull-free trigraph that is not obtained from smaller bull-free trigraphs by substitution. Then at least one of the following holds:*

- *G or $\overline{G}$ is an elementary expansion of a member of $\mathcal{T}_1$;*
- *G is an elementary expansion of a member of $\mathcal{T}_2$.*

Conversely, if H is a trigraph such that either one of H and $\overline{H}$ is an elementary expansion of a member of $\mathcal{T}_1$, or H is an elementary expansion of a trigraph in $\mathcal{T}_2$, then H is an elementary bull-free trigraph.

We note that some trigraphs H that satisfy the hypotheses of 3.3 admit a homogeneous set decomposition (i.e., they can be obtained by substitution from smaller bull-free trigraphs).

The definitions of classes $\mathcal{T}_1$ and $\mathcal{T}_2$, as well as of elementary expansions, are long and complicated, and we do not give them in this section. Instead, we give the definition of an elementary expansion of a trigraph in Section 4; we prove there that if G is an elementary expansion of a trigraph H, then G is Berge if and only if H is. In Section 5, we give the definition of the class $\mathcal{T}_1$, and we derive the class $\mathcal{T}_1^*$ of all Berge trigraphs in $\mathcal{T}_1$. In Section 6, we give the definition of the class $\mathcal{T}_2$, and we prove that every trigraph in $\mathcal{T}_2$ is Berge. In Section 7 (the final section), we put all of this together to derive the structure theorem for Berge bull-free trigraphs, which we state below.

3.4. *Let G be a trigraph. Then G is bull-free and Berge if and only if at least one of the following holds:*

- *G is obtained from smaller bull-free Berge trigraphs by substitution;*
- *G or $\overline{G}$ is an elementary expansion of a trigraph in $\mathcal{T}_1^*$;*
- *G is an elementary expansion of a trigraph in $\mathcal{T}_2$.*

4. ELEMENTARY EXPANSIONS

Our goal in this section is to prove that if a trigraph G is an elementary expansion of a trigraph H, then G is Berge if and only if H is Berge. Informally, a trigraph G is said to be an "elementary expansion" of a trigraph H provided that G can be obtained by "expanding" some semiadjacent pairs of a certain kind to homogeneous pairs of a corresponding kind. We start by defining the two kinds of semiadjacent pair and the two kinds of homogeneous pair that we will need. After that, we define elementary expansions.

Semiadjacent pairs of type one and two. Let G be a trigraph, let (a, b) be a semiadjacent pair in G, and let $(\{a\}, \{b\}, C, D, E, F)$ be the partition of G with respect to the homogeneous pair $(\{a\}, \{b\})$.

We say that (a, b) is a semiadjacent pair of *type one* provided all of the following hold:

- C, D, and F are nonempty;
- E is empty;
- neither C nor D is strongly anticomplete to F.

We say that (a, b) is a semiadjacent pair of *type two* provided all of the following hold:

- C, D, and F are nonempty;
- E is empty;
- C is not strongly anticomplete to F;
- D is strongly anticomplete to F.

Finally, a semiadjacent pair (a, b) in a trigraph G is said to be of *complement type one* or of *complement type two* in G provided that (a, b) is a semiadjacent pair of complement type one or two, respectively, in $\overline{G}$.

Closures of rooted forests. We say that a trigraph T is a *forest* provided that there are neither triangles nor holes in T. (Thus, for any two vertices of T, there is at most one path between them.) A connected forest is called a *tree*. A rooted forest is a $(k+1)$-tuple $\mathfrak{T} = (T, r_1, \ldots, r_k)$, where T is a forest with components $T_1, \ldots, T_k$ such that $r_i \in V_{T_i}$ for all $i \in \{1, \ldots, k\}$. Given distinct $u, v \in V_T$, we say that u is a *descendant* of v, or that v is an *ancestor* of u, provided that $u, v \in V_{T_i}$ for some $i \in \{1, \ldots, k\}$, and that if P is the (unique) path from u to r_i then $v \in V_P$. We say that u and v are *comparable* in $\mathfrak{T}$ provided that u is either an ancestor or a descendant of v. We say that u is a *child* of v, or that v is the *parent* of u, provided that u and v are adjacent, and that u is a descendant of v. A vertex $v \in V_T$ is a *leaf* in $\mathfrak{T}$ provided that v has no descendants. We say that the rooted forest $\mathfrak{T}$ is *good* provided that for all semiadjacent $u, v \in V_T$, one of u and v is a leaf in $\mathfrak{T}$. Finally, we say that the trigraph T' is the *closure* of the rooted forest $\mathfrak{T} = (T, r_1, \ldots, r_k)$ provided that:

- $V_{T'} = V_T$;
- for all distinct $u, v \in V_{T'}$, uv is an adjacent pair in T' if and only if u and v are comparable in $\mathfrak{T}$;
- for all distinct $u, v \in V_{T'}$, uv is a semiadjacent pair in T' if and only if uv is a semiadjacent pair in T.

Homogeneous pairs of type one and two. A homogeneous pair (A, B) in a trigraph G is said to be of *type one* in G provided that the associated partition (A, B, C, D, E, F) of G satisfies all of the following:

(1) A is neither strongly complete nor strongly anticomplete to B;
(2) $3 \leq |A \cup B| \leq |V_G| - 3$;
(3) A and B are strongly stable sets;
(4) C, D, and F are all nonempty;
(5) E is empty;
(6) neither C nor D is strongly anticomplete to F.

A homogeneous pair (A, B) in a trigraph G is said to be of *type two* in G provided there exists a good rooted forest $\mathfrak{T} = (T, r_1, \ldots, r_k)$ such that the partition (A, B, C, D, E, F) of G associated with (A, B) satisfies all of the following:

(1) A is neither strongly complete nor strongly anticomplete to B;
(2) $3 \leq |A \cup B| \leq |V_G| - 3$;
(3) A is a strongly stable set;
(4) $G[B]$ is the closure of $\mathfrak{T}$;
(5) if $a \in A$ is adjacent to $b \in B$, then a is strongly adjacent to every descendant of b in $\mathfrak{T}$;
(6) if all of the following hold:
  - $u, v \in B$ and $u, v \in V_{T_i}$ for some $i \in \{1, \ldots, k\}$,
  - u is a child of v in $\mathfrak{T}$,
  - P is the (unique) path in T_i between r_i and v,
  - X is the component of $T_i \setminus (V_P \setminus \{v\})$ that contains u and v,
  - Y is the set of vertices of X that are semiadjacent to v,
  - $a \in A$ is adjacent to u and antiadjacent to v;

  then a is strongly complete to Y and to $B \setminus (V_X \cup V_P)$, and strongly anticomplete to $V_P \setminus \{v\}$;
(7) C, D, and F are all nonempty;
(8) E is empty;
(9) C is not strongly anticomplete to F;
(10) D is strongly anticomplete to F.

We will need the following result.

4.1. *Let G be a trigraph, and let (A, B) be a homogeneous pair of type one or two in one of G and $\overline{G}$. Then (A, B) is a good homogeneous pair in G.*

Proof. Recall that (A, B) is a good homogeneous pair in G if and only if (A, B) is a good homogeneous pair in $\overline{G}$. So we may assume that (A, B) is a homogeneous pair of type one or two in G. Now, we need to prove the following:

- neither $G[A]$ nor $G[B]$ contains a three-edge path;
- there does not exist a path $v_1 - v_2 - v_3 - v_4$ in G such that $v_1, v_4 \in A$ and $v_2, v_3 \in B$;
- there does not exist a path $v_1 - v_2 - v_3 - v_4$ in G such that $v_1, v_4 \in B$ and $v_2, v_3 \in A$.

If (A, B) is a homogeneous pair of type one, then A and B are both stable, and the result is immediate. So assume that (A, B) is a homogeneous pair of type two. Then A is stable, and so $G[A]$ contains no three-edge path. Furthermore, there is no path $v_1 - v_2 - v_3 - v_4$ in G with $v_1, v_4 \in B$ and $v_2, v_3 \in A$. Let $\mathfrak{T} = (T, r_1, \ldots, r_k)$ be a good rooted forest such that $G[B]$ is the closure of $\mathfrak{T}$, as in the definition of a homogeneous pair of type two.

Suppose that $v_1 - v_2 - v_3 - v_4$ is a three-edge path in $G[B]$; then $v_1, v_2, v_3, v_4 \in V_{T_i}$ for some component T_i of T. Since $v_1 - v_2 - v_3$ is a path, v_2 is comparable to both v_1 and v_3 in $\mathfrak{T}$, and either v_1 and v_3 are not comparable in $\mathfrak{T}$ or there exist distinct $i, j \in \{1, 2\}$ such that v_i is a leaf in $\mathfrak{T}$ and v_i is a child of and is semiadjacent to v_j; it then easily follows that v_2 is an ancestor of both v_1 and v_3. Similarly, since $v_2 - v_3 - v_4$ is a path, v_3 is an ancestor of both v_2 and v_4. But then v_2 is an ancestor of v_3, and v_3 is an ancestor of v_2, which is impossible. Thus, $G[B]$ contains no three-edge path.

Suppose now that $v_1 - v_2 - v_3 - v_4$ is a three-edge path in G with $v_1, v_4 \in A$ and $v_2, v_3 \in B$. Then v_2 and v_3 are comparable in $\mathfrak{T}$; by symmetry, we may assume that v_3 is a descendant of v_2. But then the fact that v_1 is adjacent to v_2 implies that v_1 is strongly adjacent to v_3, which contradicts the fact that $v_1 - v_2 - v_3 - v_4$ is a path. ■

We now give the definition of an elementary expansion of a trigraph, and prove the main result of this section.

Elementary expansions. Let H and G be trigraphs. We say that G is an *elementary expansion* of H provided that $V_G = \bigcup_{v \in V_H} X_v$, where the X_v's are nonempty and pairwise disjoint, and all of the following hold:

(1) if $u, v \in V_H$ are strongly adjacent, then X_u is strongly complete to X_v;
(2) if $u, v \in V_H$ are strongly antiadjacent, then X_u is strongly anticomplete to X_v;
(3) if $v \in V_H$ is not an endpoint of any semiadjacent pair of type one or two, or of complement type one or two, then $|X_v| = 1$;
(4) if $u, v \in V_H$ are semiadjacent, and neither (u, v) nor (v, u) is a semiadjacent pair of type one or two, or of complement type one or two, then the unique vertex of X_u is semiadjacent to the unique vertex of X_v;
(5) if (u, v) is a semiadjacent pair of type one or two in H, then either $|X_u| = |X_v| = 1$ and the unique vertex of X_u is semiadjacent to the unique vertex of X_v, or (X_u, X_v) is a homogeneous pair of type one or two, respectively, in G;
(6) if (u, v) is a semiadjacent pair of complement type one or two in H, then either $|X_u| = |X_v| = 1$ and the unique vertex of X_u is semiadjacent to the unique vertex of X_v, or (X_u, X_v) is a homogeneous pair of type one or two, respectively, in $\overline{G}$;

Note that every trigraph is an elementary expansion of itself.

4.2. *Let G and H be trigraphs, and assume that G is an elementary expansion of H. Then G is Berge if and only if H is Berge.*

Proof. The "only if" part follows from the fact that every realization of H is an induced subgraph of some realization of G. To prove the "if" part, we assume that H is Berge. Suppose that G is not Berge, and let W be the vertex set of an odd hole or an odd antihole in G. By 4.1 and 2.3, we have that $|W \cap X_v| \leq 1$ for all $v \in V_H$. But then $\{v \in V_H \mid W \cap X_v \neq \emptyset\}$ is the vertex set of an odd hole or an odd antihole in H, which contradicts the assumption that H is Berge. ■

5. CLASS $\mathcal{T}_1$

In this section, we state the definition of the class $\mathcal{T}_1$ from [5], and we derive the class $\mathcal{T}_1^*$ of all Berge trigraphs in $\mathcal{T}_1$. The section is organized as follows. We first define "clique connectors" and "tulips." Clique connectors can conveniently be thought of as the basic "building blocks" of trigraphs in $\mathcal{T}_1$ and $\mathcal{T}_1^*$. A clique connector consists of a bipartite trigraph and a strong clique that "attaches" to the bipartite trigraph in a certain specified way; a tulip is a special kind of clique connector. We next introduce trigraphs called "tulip beds," which consist of a bipartite trigraph and an unlimited number of strong cliques that "attach" to the bipartite trigraph as partially overlapping tulips. We prove that each tulip bed is Berge (see 5.6). We then define "melts" (which are tulip beds and therefore Berge), and trigraphs that "admit an H-structure" for some "usable" graph H. The class

$\mathcal{T}_1$ is defined to be the collection of all melts and all trigraphs that admit an H-structure for some usable graph H. Finally, we define the subclass $\mathcal{T}_1^*$ of $\mathcal{T}_1$, and to complete the section, we prove that every trigraph in $\mathcal{T}_1^*$ is a tulip bed (and therefore Berge), and that every Berge trigraph in $\mathcal{T}_1$ is in $\mathcal{T}_1^*$. (However, we note that not every tulip bed is bull-free, and consequently, the class $\mathcal{T}_1^*$ is only a proper subclass of the class of all tulip beds.)

Clique connectors. Let G be a trigraph such that $V_G = K \cup A \cup B \cup C \cup D$, where K, A, B, C, and D are pairwise disjoint. Assume that $K = \{k_1, \dots, k_t\}$ is a strong clique, and that A, B, C, and D are strongly stable sets. Let $A = \bigcup_{i=1}^t A_i$, $B = \bigcup_{i=1}^t B_i$, $C = \bigcup_{i=1}^t C_i$, and $D = \bigcup_{i=1}^t D_t$, and assume that $A_1, \dots, A_t$, $B_1, \dots, B_t$, $C_1, \dots, C_t$, $D_1, \dots, D_t$ are pairwise disjoint. Assume that for all $i \in \{1, \dots, t\}$, the following hold:

(1) A_i is strongly complete to $\{k_1, \dots, k_{i-1}\}$;
(2) A_i is complete to $\{k_i\}$;
(3) A_i is strongly anticomplete to $\{k_{i+1}, \dots, k_t\}$;
(4) B_i is strongly complete to $\{k_{t-i+2}, \dots, k_t\}$;
(5) B_i is complete to $\{k_{t-i+1}\}$;
(6) B_i is strongly anticomplete to $\{k_1, \dots, k_{t-i}\}$.

For each $i \in \{1, \dots, t\}$, let A'_i be the set of all vertices in A_i that are semiadjacent to k_i, and let B'_i be the set of all vertices in B_i that are semiadjacent to k_{t-i+1} (thus, $|A'_i| \leq 1$ and $|B'_i| \leq 1$). Next, assume that:

(7) if there exist some $i, j \in \{1, \dots, t\}$ such that $i + j \neq t$ and A_i is not strongly complete to B_j, then $|K| = |A| = |B| = 1$, and the unique vertex of A is semiadjacent to the unique vertex of B;
(8) for all $i \in \{1, \dots, t\}$, A'_i is strongly complete to B_{t-i}, B'_{t-i} is strongly complete to A_i, and the adjacency between $A_i \setminus A'_i$ and $B_{t-i} \setminus B'_{t-i}$ is arbitrary;
(9) $A \cup K$ is strongly anticomplete to D, and $B \cup K$ is strongly anticomplete to C;
(10) for all $i \in \{1, \dots, t\}$, C_i is strongly complete to $\bigcup_{j=1}^{i-1} A_j$ and strongly anticomplete to $\bigcup_{j=i+1}^{t} A_j$;
(11) for all $i \in \{1, \dots, t\}$, C_i is strongly complete to A'_i, every vertex of C_i has a neighbor in A_i, and otherwise the adjacency between C_i and $A_i \setminus A'_i$ is arbitrary;
(12) for all $i \in \{1, \dots, t\}$, D_i is strongly complete to $\bigcup_{j=1}^{i-1} B_j$ and strongly anticomplete to $\bigcup_{j=i+1}^{t} B_j$;
(13) for all $i \in \{1, \dots, t\}$, D_i is strongly complete to B'_i, every vertex of D_i has a neighbor in B_i, and otherwise the adjacency between D_i and $B_i \setminus B'_i$ is arbitrary;
(14) for all $i, j \in \{1, \dots, t\}$, if $i + j > t$ then C_i is strongly complete to D_j, and otherwise the adjacency between C_i and D_j is arbitrary;
(15) A_t and B_t are both nonempty.

We then say that G is a (K, A, B, C, D)-*clique connector*. If for all $i, j \in \{1, \dots, t\}$ such that $i + j \neq t$ we have that A_i is strongly complete to B_j, then we say that G is a *nondegenerate* (K, A, B, C, D)-clique connector; otherwise, we say that G is *degenerate*. If C and D are both empty, and for all $i, j \in \{1, \dots, t\}$ such that $i + j \neq t$ we have that A_i is strongly complete to B_j, then we say that G is a (K, A, B)-*tulip*.

We say that a trigraph G is a *clique connector* provided that there exist some K, A, B, C, D such that G is a (K, A, B, C, D)-clique connector; we say that G is a *degenerate* (respectively: *nondegenerate*) clique connector provided that there exist some

K, A, B, C, D such that G is a degenerate (respectively: nondegenerate) (K, A, B, C, D)-clique connector. We say that G is a *tulip* if there exist some K, A, B such that G is a (K, A, B)-tulip.

We observe that G is a (K, A, B, C, D)-clique connector if and only if G is a (K, B, A, D, C)-clique connector; similarly, G is a (K, A, B)-tulip if and only if G is a (K, B, A)-tulip; we will exploit this symmetry throughout the section. We also note that if G is a (K, A, B, C, D)-clique connector, then $G[A \cup B \cup C \cup D]$ is a bipartite trigraph with bipartition $(A \cup D, B \cup C)$. Finally, we note that G is a (K, A, B)-tulip if and only if G is a nondegenerate $(K, A, B, \emptyset, \emptyset)$-clique connector.

All nondegenerate clique connectors (and therefore, all tulips) are Berge, as we will see in a slightly more general setting later in the section (see 5.6 and the comment after it). For now, we prove three results about clique connectors and tulips. The first (5.1) gives a necessary and sufficient condition for a degenerate clique connector to be Berge; the second (5.2) states that each Berge degenerate clique connector becomes nondegenerate after relabeling; and the third (5.3) is a technical lemma about tulips that will be used throughout this section.

5.1. *Let G be a degenerate (K, A, B, C, D)-clique connector. Then G is Berge if and only if at least one of C and D is empty.*

Proof. Since G is degenerate, we can set $K = \{k_1\}$, $A = A_1 = \{a\}$, and $B = B_1 = \{b\}$, with a and b semiadjacent. Furthermore, by axiom (14) from the definition of a clique connector, we know that C is strongly complete to D. Now, for the "only if" part, we observe that if both C and D are nonempty with some $c \in C$ and $d \in D$, then $k_1 - a - c - d - b - k_1$ is an odd hole in G, and so G is not Berge. For the "if" part, suppose that at least one of C and D is empty. If both C and D are empty, then $|V_G| = 3$ and G is Berge. So suppose that exactly one of C and D is empty; by symmetry, we may assume that $C \neq \emptyset$ and $D = \emptyset$. Now, we claim that C is a homogeneous set in G. First, we know by axiom (11) from the definition of a (K, A, B, C, D)-clique connector that every vertex in C has a neighbor in A; since $A = \{a\}$ and a is semiadjacent to $b \notin C$, it follows that C is strongly complete to A. Second, by axiom (9), we know that C is strongly anticomplete to $K \cup B$. Thus, C is a homogeneous set in G, as claimed. Since $|K| = |A| = |B| = 1$, and since $D = \emptyset$, it follows that G is obtained by substituting the trigraph $G[C]$ for a vertex in a four-vertex trigraph. $G[C]$ is Berge because C is a strongly stable set in G, and clearly, every four-vertex trigraph is Berge. By 2.2 then, G is Berge. ■

5.2. *If G is a degenerate $(K, A, B, C, \emptyset)$-clique connector, then G is a nondegenerate $(B, A, K, C, \emptyset)$-clique connector, and if G is a degenerate $(K, A, B, \emptyset, D)$-clique connector, then G is a nondegenerate $(A, K, B, \emptyset, D)$-clique connector.*

Proof. This is immediate from the definition. ■

5.3. *Let G be a (K, A, B)-tulip, and let $p_1 - p_2 - p_3 - p_4$ be a path in G such that $p_2, p_3 \in K$. Then either $p_1 \in A$ and $p_4 \in B$, or $p_1 \in B$ and $p_4 \in A$.*

Proof. Since K is a strong clique, we know that $p_1, p_4 \notin K$; thus, $p_1, p_4 \in A \cup B$. Now, suppose that neither of the stated outcomes holds. By symmetry then, we may assume that $p_1, p_4 \in A$. Set $K = \{k_1, \ldots, k_t\}$ as in the definition of a tulip, and set

$p_2 = k_i$ and $p_3 = k_j$; by symmetry, we may assume that $i < j$. Since p_4 is adjacent to $p_3 = k_j$, we know that p_4 is strongly complete to $\{k_1, \dots, k_{j-1}\}$, and so in particular, p_4 is strongly adjacent to $p_2 = k_i$, which is a contradiction. ■

Tulip beds. We say that a trigraph G is a *tulip bed* provided that either G is bipartite, or V_G can be partitioned into (nonempty) sets $F_1, F_2, Y_1, \dots, Y_s$ (for some integer $s \geq 1$) such that all of the following hold:

(1) F_1 and F_2 are strongly stable sets;
(2) $Y_1, \dots, Y_s$ are strong cliques, pairwise strongly anticomplete to each other;
(3) for all $v \in F_1 \cup F_2$, v has neighbors in at most two of $Y_1, \dots, Y_s$;
(4) for all adjacent $v_1 \in F_1$ and $v_2 \in F_2$, v_1 and v_2 have common neighbors in at most one of $Y_1, \dots, Y_s$;
(5) for all $l \in \{1, \dots, s\}$, if X_l is the set of all vertices in $F_1 \cup F_2$ with a neighbor in Y_l, then $G[Y_l \cup X_l]$ is a $(Y_l, X_l \cap F_1, X_l \cap F_2)$-tulip.

As we stated at the beginning of this section, not all tulip beds are bull-free (e.g., a bull that contains no semiadjacent pairs is easily seen to be a tulip bed). However, all tulip beds are Berge, and we now turn to proving this fact. We begin with some technical lemmas.

5.4. *Let G be a tulip bed, and let $F_1, F_2, Y_1, \dots, Y_s, X_1, \dots, X_s$ be as in the definition of a tulip bed. Let $v_1 \in F_1$, $v_2 \in F_2$, and $l \in \{1, \dots, s\}$, and assume that v_1 and v_2 have a common neighbor in Y_l. Then both of the following hold:*

- *v_1v_2 is a strongly adjacent pair;*
- *v_1 and v_2 have no common antineighbor in Y_l.*

Proof. Clearly, $v_1, v_2 \in X_l$. Set $K = Y_l$, $A = X_l \cap F_1$, and $B = X_l \cap F_2$. Now $G[Y_l \cup X_l]$ is a (K, A, B)-tulip, $v_1 \in A$, $v_2 \in B$, and v_1 and v_2 have a common neighbor in K. Set $K = \{k_1, \dots, k_t\}$, $A = \bigcup_{i=1}^{t} A_i$, and $B = \bigcup_{i=1}^{t} B_i$ as in the definition of a (K, A, B)-tulip. Fix $i \in \{1, \dots, t\}$ such that k_i is a common neighbor of v_1 and v_2. Fix $p, q \in \{1, \dots, t\}$ such that $v_1 \in A_p$ and $v_2 \in B_q$. Since $v_1 \in A_p$ is adjacent to k_i, we know by axioms (1), (2), and (3) from the definition of a (K, A, B)-tulip that $i \leq p$; and since $v_2 \in B_q$ is adjacent to k_i, we know by axioms (4), (5), and (6) that $t - q + 1 \leq i$. Thus, $t - q + 1 \leq p$, and so $p + q \geq t + 1$. In particular, $p + q \neq t$, and so A_p is strongly complete to B_q (this follows from the fact that tulips are nondegenerate clique connectors). Since $v_1 \in A_p$ and $v_2 \in B_q$, it follows that v_1v_2 is a strongly adjacent pair.

It remains to show that v_1 and v_2 do not have a common antineighbor in Y_l. Suppose otherwise; fix $j \in \{1, \dots, t\}$ such that k_j is antiadjacent to both v_1 and v_2. Since k_j is antiadjacent to $v_1 \in A_p$, we know by axioms (1), (2), and (3) from the definition of a (K, A, B)-tulip that $p \leq j$; and since k_j is antiadjacent to $v_2 \in B_q$, we know by axioms (4), (5), and (6) that $j \leq t - q + 1$. But now $p \leq t - q + 1$, and so $p + q \leq t + 1$. We showed before that $p + q \geq t + 1$, and so it follows that $p + q = t + 1$, and consequently, that $j = p = t - q + 1$. Since $k_j = k_p$ is antiadjacent to $v_1 \in A_p$, axiom (2) from the definition of a (K, A, B)-tulip implies that k_j is semiadjacent to v_1; similarly, since $k_j = k_{t-q+1}$ is antiadjacent to $v_2 \in B_q$, axiom (5) implies that k_j is semiadjacent to v_2. But now k_j is semiadjacent to both v_1 and v_2, which is impossible by the definition of a trigraph. ■

We remark that a result very similar to 5.4 was proven in [3] (see the proof of 3.1, statements (1) and (3), from [3]).

5.5. *No tulip bed contains a three-edge path with a center.*

Proof. Let G be a tulip bed, and suppose that $p_1 - p_2 - p_3 - p_4$ is a path with a center p_c in G. Since G contains a triangle, G is not bipartite. Then let $F_1, F_2, Y_1, \ldots, Y_s, X_1, \ldots, X_s$ be as in the definition of a tulip bed.

Our first goal is to show that $p_c \notin F_1 \cup F_2$. Suppose otherwise. Since $\{p_c, p_2, p_3\}$ is a triangle, we know that p_2 and p_3 cannot both lie in $F_1 \cup F_2$; by symmetry, we may assume that $p_2 \notin F_1 \cup F_2$; thus, $p_2 \in Y_l$ for some $l \in \{1, \ldots, s\}$. We claim that $p_3 \in Y_l$. Suppose otherwise. Since p_2p_3 is an adjacent pair and the strong cliques $Y_1, \ldots, Y_s$ are strongly anticomplete to each other, this means that $p_3 \in F_1 \cup F_2$. Now, $\{p_c, p_3, p_4\}$ is a triangle, and so $p_4 \notin F_1 \cup F_2$. Since $p_c, p_3 \in F_1 \cup F_2$ are adjacent with a common neighbor $p_2 \in Y_l$, axiom (4) from the definition of a tulip bed implies that all common neighbors of p_c and p_3 lie in Y_l, and so $p_4 \in Y_l$. But then $p_2, p_4 \in Y_l$, which is impossible since p_2p_4 is an antiadjacent pair and Y_l is a strong clique. Thus, $p_3 \in Y_l$. Now, $p_2, p_3 \in Y_l$, Y_l is a strong clique, and p_1p_3 and p_2p_4 are antiadjacent pairs; thus, $p_1, p_4 \notin Y_l$, and therefore, $p_1, p_4 \in F_1 \cup F_2$. Clearly, $p_c, p_1, p_4 \in X_l$; by symmetry, we may assume that $p_c \in X_l \cap F_1$. Since p_c is complete to $\{p_1, p_4\}$ and F_1 is strongly stable, it follows that $p_1, p_4 \in X_l \cap F_2$. But then the path $p_1 - p_2 - p_3 - p_4$ contradicts 5.3. This proves that $p_c \notin F_1 \cup F_2$.

Let $l \in \{1, \ldots, s\}$ be such that $p_c \in Y_l$. Since $p_c \in Y_l$ is complete to $\{p_1, p_2, p_3, p_4\}$, we know that $p_1, p_2, p_3, p_4 \in Y_l \cup X_l$. Since p_1p_4 is an antiadjacent pair, p_1 and p_4 cannot both lie in Y_l; by symmetry, we may assume that $p_1 \in X_l \cap F_1$. Since p_1p_2 is an adjacent pair, there are two cases to consider: when $p_2 \in Y_l$, and when $p_2 \in X_l \cap F_2$. Suppose first that $p_2 \in Y_l$. Since p_2p_4 is an antiadjacent pair, this means that $p_4 \notin Y_l$; since p_1p_4 is an antiadjacent pair with a common neighbor in Y_l, and since $p_1 \in X_l \cap F_1$, 5.4 implies that $p_4 \in X_l \cap F_1$. Since p_3p_4 is an adjacent pair, we know that $p_3 \notin X_l \cap F_1$; and since $p_1 \in X_l \cap F_1$ and p_3 are antiadjacent with a common neighbor in Y_l, we know by 5.4 that $p_3 \notin X_l \cap F_2$. Thus, $p_3 \in Y_l$. But then the path $p_1 - p_2 - p_3 - p_4$ contradicts 5.3. Thus, $p_2 \in X_l \cap F_2$. The fact that p_1p_4 is an antiadjacent pair with a common neighbor $p_c \in Y_l$, together with the fact that $p_1 \in X_l \cap F_1$, implies (by 5.4) that $p_4 \notin X_l \cap F_2$. Similarly, since p_2p_4 is an antiadjacent pair with a common neighbor $p_c \in Y_l$, and since $p_2 \in X_l \cap F_2$, we have that $p_4 \notin X_l \cap F_1$. Finally, since $p_1 \in X_l \cap F_1$ and $p_2 \in X_l \cap F_2$ have a common neighbor $p_c \in Y_l$, 5.4 implies that p_1 and p_2 have no common antineighbor in Y_l, and so $p_4 \notin Y_l$. But then $p_4 \notin Y_l \cup X_l$, which is a contradiction. ■

5.6. *Each tulip bed is Berge.*

Proof. Let G be a tulip bed. Since every antihole of length at least seven contains a three-edge path with a center, 5.5 implies that G contains no antihole of length at least seven. Since each antihole of length five is also a hole of length five, this reduces our problem to proving that G contains no odd holes. If G is bipartite, then the result is immediate; so assume that G is not bipartite. Now let $F_1, F_2, Y_1, \ldots, Y_s, X_1, \ldots, X_s$ be as in the definition of a tulip bed.

Suppose that $w_0 - w_1 - \cdots - w_{2k} - w_0$ (with indices in $\mathbb{Z}_{2k+1}$ for some integer $k \geq 2$) is an odd hole in G, and set $W = \{w_0, w_1, \ldots, w_{2k}\}$. We will obtain a contradiction by showing that $G[W]$ is bipartite. Note that it suffices to show that for all $l \in \{1, \ldots, s\}$,

$G[W \cap (Y_l \cup F_1 \cup F_2)]$ is bipartite with some bipartition (F_1^l, F_2^l) such that $W \cap F_1 \subseteq F_1^l$ and $W \cap F_2 \subseteq F_2^l$, for then the fact that $Y_1, \dots, Y_s$ are pairwise strongly anticomplete to each other will imply that $G[W]$ is bipartite with bipartition $(\bigcup_{l=1}^{s} F_1^l, \bigcup_{l=1}^{s} F_2^l)$.

We begin by showing that for all $l \in \{1, \dots, s\}$ and $i \in \mathbb{Z}_{2k+1}$ such that $w_i \in Y_l$, w_i is strongly anticomplete to at least one of $W \cap F_1$ and $W \cap F_2$. Suppose otherwise. Fix some $l \in \{1, \dots, s\}$ and $i \in \mathbb{Z}_{2k+1}$ such that $w_i \in Y_l$, and w_i has neighbors in both $W \cap F_1$ and $W \cap F_2$. First, note that w_i is anticomplete to at least one of $W \cap F_1$ and $W \cap F_2$; indeed if there existed some $i_1, i_2 \in \mathbb{Z}_{2k+1}$ such that $w_{i_1} \in W \cap F_1$, $w_{i_2} \in W \cap F_2$, and w_i is strongly adjacent to both w_{i_1} and w_{i_2}, then (by 5.4) w_{i_1} and w_{i_2} would be strongly adjacent, and $\{w_i, w_{i_1}, w_{i_2}\}$ would be a strong triangle in $G[W]$, which is impossible. Now suppose that there exist some $i_1, i_2 \in \mathbb{Z}_{2k+1}$ such that $w_{i_1} \in W \cap F_1$, $w_{i_2} \in W \cap F_2$, w_i is adjacent to both w_{i_1} and w_{i_2} and semiadjacent to one of them. By symmetry, we may assume that w_i is strongly adjacent to w_{i_1} and semiadjacent to w_{i_2}; thus, w_i is anticomplete to $W \cap F_2$. Since $w_{i_1} \in F_1$ and $w_{i_2} \in F_2$ have a common neighbor $w_i \in Y_l$, 5.4 implies that $w_{i_1}w_{i_2}$ is a strongly adjacent pair. Since $w_iw_{i_1}$ and $w_{i_1}w_{i_2}$ are strongly adjacent pairs, by symmetry, we may assume that $w_{i_1} = w_{i+1}$ and $w_{i_2} = w_{i+2}$. Since w_iw_{i+2} is a semiadjacent pair, $w_{i-1}w_i$ is a strongly adjacent pair; as w_i is anticomplete to $W \cap F_2$, it follows that $w_{i-1} \notin F_2$. Next, the fact that $w_{i+1} \in F_1$ and $w_{i+2} \in F_2$ have a common neighbor in Y_l implies (by 5.4) that w_{i+1} and w_{i+2} do not have a common antineighbor in Y_l; thus, the fact that w_{i-1} is antiadjacent to both w_{i+1} and w_{i+2} implies that $w_{i-1} \notin Y_l$. It follows that $w_{i-1} \in F_1$. Then since $w_{i-1} \in F_1$ and $w_{i+2} \in F_2$ have a common neighbor $w_i \in Y_l$, we know (by 5.4) that $w_{i-1}w_{i+2}$ is a strongly adjacent pair, which is impossible. Thus, w_i is strongly anticomplete to at least one of $W \cap F_1$ and $W \cap F_2$.

Next, fix $l \in \{1, \dots, s\}$. We need to show that $G[W \cap (Y_l \cup F_1 \cup F_2)]$ is bipartite with some bipartition (F_1^l, F_2^l) such that $W \cap F_1 \subseteq F_1^l$ and $W \cap F_2 \subseteq F_2^l$. Since Y_l is a strong clique, we know that $|W \cap Y_l| \leq 2$. If $W \cap Y_l = \emptyset$, then $G[W \cap (Y_l \cup F_1 \cup F_2)]$ is bipartite with bipartition $(W \cap F_1, W \cap F_2)$, and we are done. So assume that $1 \leq |W \cap Y_l| \leq 2$.

Suppose first that $|W \cap Y_l| = 1$, say $W \cap Y_l = \{w_i\}$. By the above, w_i is strongly anticomplete to at least one of $W \cap F_1$ and $W \cap F_2$. By symmetry, we may assume that w_i is strongly anticomplete to $W \cap F_1$. But then $G[W \cap (Y_l \cup F_1 \cup F_2)]$ is bipartite with bipartition $((W \cap F_1) \cup \{w_i\}, W \cap F_2)$.

Suppose now that $|W \cap Y_l| = 2$; since Y_l is a strong clique, this means that $W \cap Y_l = \{w_i, w_{i+1}\}$ for some $i \in \mathbb{Z}_{2k+1}$. Clearly then, $w_{i-1}, w_{i+2} \in X_l$. Now, $w_{i-1} - w_i - w_{i+1} - w_{i+2}$ is a three-edge path with $w_i, w_{i+1} \in Y_l$ and $w_{i-1}, w_{i+2} \in F_1 \cup F_2$; it then follows from 5.3 that either $w_{i-1} \in F_1$ and $w_{i+2} \in F_2$, or $w_{i-1} \in F_2$ and $w_{i+2} \in F_1$; by symmetry, we may assume that the former holds. Then since each of w_i and w_{i+1} is strongly anticomplete to at least one of $W \cap F_1$ and $W \cap F_2$, it follows that w_i is strongly anticomplete to $W \cap F_2$, and w_{i+1} is strongly anticomplete to $W \cap F_1$. Thus, $F[W \cap (Y_l \cup F_1 \cup F_2)]$ is bipartite with bipartition $((W \cap F_1) \cup \{w_{i+1}\}, (W \cap F_2) \cup \{w_i\})$. This completes the argument. ■

We observe that every tulip and every nondegenerate clique-connector is a tulip bed and therefore Berge.

Melts. Let G be a trigraph. Assume that $V_G = K \cup M \cup A \cup B$, where K and M are strong cliques, A and B are strongly stable sets, and K, M, A, and B are pairwise disjoint. Assume that $|A| \geq 2$ and $|B| \geq 2$, and that $K = \{k_1, \dots, k_m\}$ and $M = \{m_1, \dots, m_n\}$. Let $A = \bigcup_{i=0}^{m} \bigcup_{j=0}^{n} A_{i,j}$, where the $A_{i,j}$'s are pairwise disjoint; and let $B = \bigcup_{i=0}^{m} \bigcup_{j=0}^{n} B_{i,j}$, where the $B_{i,j}$'s are pairwise disjoint. Assume that $A_{0,0} = B_{0,0} = \emptyset$. Assume that for all $i \in \{1, \dots, m\}$, $A_{i,0} = \bigcup_{j=0}^{n} A_{i,0}^{j}$, where the $A_{i,0}^{j}$'s are pairwise disjoint, and assume that for all $j \in \{1, \dots, n\}$, $A_{0,j} = \bigcup_{i=0}^{m} A_{0,j}^{i}$, where the $A_{0,j}^{i}$'s are pairwise disjoint. Similarly, assume that for all $i \in \{1, \dots, m\}$, $B_{i,0} = \bigcup_{j=0}^{n} B_{i,0}^{j}$, where the $B_{i,0}^{j}$'s are pairwise disjoint, and assume that for all $j \in \{1, \dots, n\}$, $B_{0,j} = \bigcup_{i=0}^{m} B_{0,j}^{i}$, where the $B_{0,j}^{i}$'s are pairwise disjoint. Assume also that:

(1) K is strongly anticomplete to M;
(2) for all $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$, $A_{i,j}$ is:
  - strongly complete to $\{k_1, \dots, k_{i-1}\} \cup \{m_{n-j+2}, \dots, m_n\}$,
  - complete to $\{k_i, m_{n-j+1}\}$,
  - strongly anticomplete to $\{k_{i+1}, \dots, k_m\} \cup \{m_1, \dots, m_{n-j}\}$;
(3) for all $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$, $B_{i,j}$ is:
  - strongly complete to $\{k_{m-i+2}, \dots, k_m\} \cup \{m_1, \dots, m_{j-1}\}$,
  - complete to $\{k_{m-i+1}, m_j\}$,
  - strongly anticomplete to $\{k_1, \dots, k_{m-i}\} \cup \{m_{j+1}, \dots, m_n\}$;
(4) for all $i \in \{1, \dots, m\}$, $A_{i,0}$ is:
  - strongly complete to $\{k_1, \dots, k_{i-1}\}$,
  - complete to $\{k_i\}$,
  - strongly anticomplete to $\{k_{i+1}, \dots, k_m\} \cup M$;
(5) for all $j \in \{1, \dots, n\}$, $A_{0,j}$ is:
  - strongly complete to $\{m_{n-j+2}, \dots, m_n\}$,
  - complete to $\{m_{n-j+1}\}$,
  - strongly anticomplete to $K \cup \{m_1, \dots, m_{n-j}\}$;
(6) for all $i \in \{1, \dots, m\}$, $B_{i,0}$ is:
  - strongly complete to $\{k_{m-i+2}, \dots, k_m\}$,
  - complete to $\{k_{m-i+1}\}$,
  - strongly anticomplete to $\{k_1, \dots, k_{m-i}\} \cup M$;
(7) for all $j \in \{1, \dots, n\}$, $B_{0,j}$ is:
  - strongly complete to $\{m_1, \dots, m_{j-1}\}$,
  - complete to $\{m_j\}$,
  - strongly anticomplete to $K \cup \{m_{j+1}, \dots, m_n\}$;
(8) the sets $\bigcup_{j=0}^{n} A_{m,j}$, $\bigcup_{i=0}^{m} A_{i,n}$, $\bigcup_{j=0}^{n} B_{m,j}$, and $\bigcup_{i=0}^{m} B_{i,n}$ are all nonempty;
(9) for all $i, i' \in \{0, \dots, m\}$ and $j, j' \in \{0, \dots, n\}$ such that $i < i'$ and $j < j'$, at least one of the sets $A_{i,j}$ and $A_{i',j'}$ is empty, and at least one of the sets $B_{i,j}$ and $B_{i',j'}$ is empty;
(10) for all $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$, $A_{i,j}$ is strongly complete to B, and $B_{i,j}$ is strongly complete to A;
(11) for all $i, i' \in \{1, \dots, m\}$, $A_{i,0}$ is strongly complete to $B_{i',0}$;
(12) for all $j, j' \in \{1, \dots, n\}$, $A_{0,j}$ is strongly complete to $B_{0,j'}$;
(13) for all $i \in \{1, \dots, m\}$, $A_{i,0}^{0}$ is strongly anticomplete to $\bigcup_{j=1}^{n} B_{0,j}$, and every vertex of $A_{i,0}^{0}$ has a neighbor in $\bigcup_{i'=1}^{m} \bigcup_{j=1}^{n} B_{i',j}$;
(14) wfor all $j \in \{1, \dots, n\}$, $A_{0,j}^{0}$ is strongly anticomplete to $\bigcup_{i=1}^{m} B_{i,0}$, and every vertex of $A_{0,j}^{0}$ has a neighbor in $\bigcup_{i=1}^{m} \bigcup_{j'=1}^{n} B_{i,j'}$;
(15) for all $i \in \{1, \dots, m\}$, $B_{i,0}^{0}$ is strongly anticomplete to $\bigcup_{j=1}^{n} A_{0,j}$, and every vertex of $B_{i,0}^{0}$ has a neighbor in $\bigcup_{i'=1}^{m} \bigcup_{j=1}^{n} A_{i',j}$;

(16) for all $j \in \{1, \dots, n\}$, $B^0_{0,j}$ is strongly anticomplete to $\bigcup_{i=1}^{n} A_{i,0}$, and every vertex of $B^0_{0,j}$ has a neighbor in $\bigcup_{i=1}^{m} \bigcup_{j'=1}^{n} A_{i,j'}$;
(17) for all $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$:
- every vertex of $A^i_{0,j}$ has a neighbor in $B_{i,0}$,
- $A^i_{0,j}$ is strongly complete to $\bigcup_{i'=1}^{i-1} B_{i',0}$,
- $A^i_{0,j}$ is strongly anticomplete to $\bigcup_{i'=i+1}^{m} B_{i',0}$,
- every vertex of $A^j_{i,0}$ has a neighbor in $B_{0,j}$,
- $A^j_{i,0}$ is strongly complete to $\bigcup_{j'=1}^{j-1} B_{0,j'}$,
- $A^j_{i,0}$ is strongly anticomplete to $\bigcup_{j'=j+1}^{n} B_{0,j'}$,
- every vertex of $B^i_{0,j}$ has a neighbor in $A_{i,0}$,
- $B^i_{0,j}$ is strongly complete to $\bigcup_{i'=1}^{i-1} A_{i',0}$,
- $B^i_{0,j}$ is strongly anticomplete to $\bigcup_{i'=i+1}^{m} A_{i',0}$,
- every vertex of $B^j_{i,0}$ has a neighbor in $A_{0,j}$,
- $B^j_{i,0}$ is strongly complete to $\bigcup_{j'=1}^{j-1} A_{0,j'}$,
- $B^j_{i,0}$ is strongly anticomplete to $\bigcup_{j'=j+1}^{n} A_{0,j'}$.

For all $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$:

(18) let $A'_{i,0}$ be the set of all vertices in $A_{i,0}$ that are semiadjacent to k_i;
(19) let $A'_{0,j}$ be the set of all vertices of $A_{0,j}$ that are semiadjacent to m_{n-j+1};
(20) let $B'_{i,0}$ be the set of all vertices of $B_{i,0}$ that are semiadjacent to k_{m-i+1};
(21) let $B'_{0,j}$ be the set of all vertices of $B_{0,j}$ that are semiadjacent to m_j.

Assume that:

(22) for all $i \in \{1, \dots, m\}$, $A'_{i,0}$ is strongly complete to $\bigcup_{j=1}^{n} B^i_{0,j}$;
(23) for all $j \in \{1, \dots, n\}$, $A'_{0,j}$ is strongly complete to $\bigcup_{i=1}^{m} B^j_{i,0}$;
(24) for all $i \in \{1, \dots, m\}$, $B'_{i,0}$ is strongly complete to $\bigcup_{j=1}^{n} A^i_{0,j}$;
(25) for all $j \in \{1, \dots, n\}$, $B'_{0,j}$ is strongly complete to $\bigcup_{i=1}^{m} A^j_{i,0}$.

Finally, assume that:

(26) there exist some $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$ such that at least one of $A_{i,j}$ and $B_{i,j}$ is nonempty;
(27) for all $i, i' \in \{1, \dots, m\}$ and $j, j' \in \{1, \dots, n\}$, if $i + i' \geq m + 1$ and $j + j' \geq n + 1$, then at least one of $A_{i,j}$ and $B_{i',j'}$ is empty.

Under these circumstances, we say that G is a *melt*. We say that G is an *A-melt* if $B_{i,j} = \emptyset$ for all $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$. We say that G is a *B-melt* if $A_{i,j} = \emptyset$ for all $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$. We say that G is a *double melt* if there exist $i, i' \in \{1, \dots, m\}$ and $j, j' \in \{1, \dots, n\}$ such that $A_{i,j} \neq \emptyset$ and $B_{i',j'} \neq \emptyset$.

5.7. *Every melt is a tulip bed, and consequently, every melt is Berge.*

Proof. Let G be a melt; we use the notation from the definition of a melt. Set $F_1 = A$, $F_2 = B$, $Y_1 = K$, and $Y_2 = M$. Further, note that $\bigcup_{i=1}^{m} \bigcup_{j=0}^{n} (A_{i,j} \cup B_{i,j})$ is the set of all vertices in $F_1 \cup F_2 = A \cup B$ with a neighbor in $Y_1 = K$; set $X_1 = \bigcup_{i=1}^{m} \bigcup_{j=0}^{n} (A_{i,j} \cup B_{i,j})$. Similarly, note that $\bigcup_{i=0}^{m} \bigcup_{j=1}^{n} (A_{i,j} \cup B_{i,j})$ is the set of all vertices in $F_1 \cup F_2 = A \cup B$

with a neighbor in $Y_2 = M$, and set $X_2 = \bigcup_{i=0}^{m} \bigcup_{j=1}^{n} (A_{i,j} \cup B_{i,j})$. With this setup, it is easy to check that G is a tulip bed, and we leave the details to the reader. Since each tulip bed is Berge (by 5.6), this implies that G is Berge. ■

In fact, it is possible to get a slightly stronger result: if G is a melt, and K, M, A, and B are as in the definition, then $G \setminus K$ and $G \setminus M$ are both nondegenerate clique connectors, as the reader can check. However, 5.7 is sufficiently strong for the purposes of this article.

The class $\mathcal{T}_1$. The *degree* of a vertex v of a graph H, denoted by $deg_H(v)$, is the number of edges of H that are incident with v; v is an *isolated vertex* in H provided that $deg_H(v) = 0$. We say that a (possibly empty) graph H is *usable* provided that H is loopless (possibly with parallel edges) and triangle-free, and that no vertex in H is of degree greater than two.

Let H be a usable graph, and let G be a trigraph. Assume that there exists some $L \subseteq V_G$ and a map

$$h : V_H \cup E_H \cup (E_H \times V_H) \to 2^{V_G \setminus L}$$

such that all of the following hold:

(1) for all distinct $x, y \in V_H \cup E_H \cup (E_H \times V_H)$, $h(x)$ and $h(y)$ are disjoint;
(2) $V_G \setminus L = \bigcup h[V_H \cup E_H \cup (E_H \times V_H)]$;
(3) for every isolated vertex $v \in V_H$, $h(v) \neq \emptyset$;
(4) for every $e \in E_H$, $h(e) \neq \emptyset$;
(5) for every $e \in E_H$ and $v \in V_H$, $h(e, v) \neq \emptyset$ if and only if e is incident with v;
(6) for all distinct $u, v \in V_H$, $h(u)$ is strongly anticomplete to $h(v)$;
(7) for all $v \in V_H$, $h(v)$ is a (possibly empty) strong clique;
(8) every vertex in L has a neighbor in at most one set $h(v)$ with $v \in V_H$;
(9) $G[L \cup \bigcup_{e \in E_H} h(e)]$ is triangle-free;
(10) for every $e \in E_H$ and $a \in L$, a is either strongly complete or strongly anticomplete to $h(e)$;
(11) for all distinct $e, f \in E_H$, $h(e)$ is either strongly complete or strongly anticomplete to $h(f)$, and if e and f share an endpoint, then $h(e)$ is strongly complete to $h(f)$;
(12) for every $e \in E_H$ and $v \in V_H$, $h(e)$ is strongly anticomplete to $h(v)$;
(13) for every $v \in V_H$, let S_v be the set of all vertices in L with a neighbor in $h(v)$, and let T_v be the set of all vertices in $(L \cup \bigcup_{e \in E_H} h(e)) \setminus S_v$ with a neighbor in S_v; then either:
   - $h(v) = \emptyset$ (in which case $S_v = T_v = \emptyset$), and we set $A_v = B_v = C_v = D_v = \emptyset$, or
   - there exist pairwise disjoint A_v, B_v, C_v, D_v such that $S_v = A_v \cup B_v$, $T_v = C_v \cup D_v$, and $G[h(v) \cup S_v \cup T_v]$ is a $(h(v), A_v, B_v, C_v, D_v)$-clique connector;
(14) for every $v \in V_H$, if $deg_H(v) \geq 1$ and $h(v) \neq \emptyset$, then $G[h(v) \cup S_v \cup T_v]$ is a non-degenerate $(h(v), A_v, B_v, C_v, D_v)$-clique connector;
(15) for all distinct $e, f \in E_H$ and $v \in V_H$, $h(e, v)$ is strongly complete to $h(f, v)$;
(16) for all (not necessarily distinct) $e, f \in E_H$ and distinct $u, v \in V_H$, $h(e, u)$ is strongly anticomplete to $h(f, v)$.
(17) for all $e \in E_H$ and $v \in V_H$, $h(e, v)$ is strongly complete to $h(v)$;
(18) for all $e \in E_H$ and distinct $u, v \in V_H$, $h(e, v)$ is strongly anticomplete to $h(u)$;
(19) for all distinct $e, f \in E_H$ and $v \in V_H$, $h(e, v)$ is strongly anticomplete to $h(f)$;

(20) for all $e \in E_H$ and $v \in V_H$, $h(e, v)$ can be partitioned into a (possibly empty) strong clique $h^c(e, v)$ and a (possibly empty) strongly stable set $h^s(e, v)$;

(21) for all $e \in E_H$ with (distinct) endpoints $u, v \in V_H$, $G[h(e) \cup h(e, v) \cup h(e, u)]$ is an $h(e)$-melt such that $h(e) = A$, $h^c(e, v) = K$, $h^c(e, u) = M$, and $h^s(e, v) \cup h^s(e, u) = B$, with $h^s(e, v) = \bigcup_{i=1}^{m} B_{i,0}$ and $h^s(e, u) = \bigcup_{j=1}^{n} B_{0,j}$, where K, M, A, B, m, and n are as in the definition of an A-melt;

(22) for all $e \in E_H$ with (distinct) endpoints $u, v \in V_H$, either all of the following hold, or they all hold with the roles of (A_u, A_v) and (B_u, B_v) switched:
  - $h(e)$ is strongly complete to $B_u \cup B_v$,
  - $h(e, v)$ is strongly complete to A_v and strongly anticomplete to $L \setminus A_v$,
  - every vertex of $(L \cup \bigcup_{f \in E_H \setminus \{e\}} h(f)) \setminus (A_u \cup A_v)$ with a neighbor in $A_u \cup A_v$ is strongly complete to $h(e)$.

We then say that G admits an *H-structure*.

We define $\mathcal{T}_1$ to be the class of all trigraphs G such that either G is a double melt or G admits an H-structure for some usable graph H. We observe that all triangle-free trigraphs are in $\mathcal{T}_1$ (a triangle-free trigraph admits an H-structure for the empty graph H), as are all clique-connectors (a clique connector admits an H-structure for the single-vertex graph H), and all melts (double melts are in $\mathcal{T}_1$ by definition, and an A-melt admits an H-structure for the complete graph H that consists of a single edge).

It is easy to see that not all trigraphs in $\mathcal{T}_1$ are Berge. First of all, if G admits an H-structure for some usable graph H, then $G[L \cup \bigcup_{e \in E_H} h(e)]$ may contain odd holes. Second, if v is an isolated vertex in H, then $G[h(v) \cup S_v \cup T_v]$ may be a degenerate clique connector, and as we saw in 5.1, not all degenerate clique connectors are Berge. It turns out that these two are the only "anomalies" whose presence can prevent a trigraph in $\mathcal{T}_1$ from being Berge. The definition of the class $\mathcal{T}_1^*$, to which we turn next, eliminates these anomalies. In addition, we note that if a trigraph G admits an H-structure for some usable graph H, and $e, f \in E_H$ are distinct edges, then $h(e)$ and $h(f)$ are strongly complete to each other if e and f share an endpoint, but the converse need not hold: $h(e)$ and $h(f)$ may be strongly complete to each other even if e and f do not share an endpoint. In the definition of $\mathcal{T}_1^*$, we use "usable 4-tuples," defined below, instead of usable graphs, in order to "encode" the adjacency in $L \cup \bigcup_{e \in E_H} h(e)$ more precisely.

The class $\mathcal{T}_1^*$. Let H be a bipartite graph (possibly empty and possibly with parallel edges), none of whose vertices are of degree greater than two. Let L be a (possibly empty) set such that $V_H \cap L = \emptyset$. Let H' be a bipartite trigraph such that $V_{H'} = E_H \cup L$. Assume that for all distinct $e, f \in E_H$ that share at least one endpoint, ef is a strongly adjacent pair in H'; assume also that every semiadjacent pair in H' has both of its endpoints in L. Let (E_1', E_2') be a bipartition of the bipartite trigraph H'. Under these circumstances, we say that (H, H', E_1', E_2') is a *usable 4-tuple*.

Let (H, H', E_1', E_2') be a usable 4-tuple, let $L = V_{H'} \setminus E_H$, let G be a trigraph, and let $E_1, E_2 \subseteq V_G$. We then say that (G, E_1, E_2) admits an (H, H', E_1', E_2')-structure provided that $L \subseteq V_G$, and that there exists a map

$$h : V_H \cup E_H \cup (E_H \times V_H) \to 2^{V_G \setminus L}$$

such that all of the following hold:

(1) for all distinct $x, y \in V_H \cup E_H \cup (E_H \times V_H)$, $h(x)$ and $h(y)$ are disjoint;

(2) $V_G = L \cup \bigcup_{v \in V_H} h(v) \cup \bigcup_{e \in E_H} h(e) \cup \bigcup_{(e,v) \in E_H \times V_H} h(e, v)$;

(3) for all $v \in V_H$, $h(v)$ is a (possibly empty) clique;
(4) for all isolated vertices $v \in V_H$, $h(v) \neq \emptyset$;
(5) for all $e \in E_H$, $h(e)$ is a (nonempty) strongly stable set;
(6) for all $e \in E_H$ and $v \in V_H$, $h(e, v) \neq \emptyset$ if and only if e is incident with v;
(7) for all $e \in E_H$ and $v \in V_H$, $h(e, v)$ can be partitioned into a (possibly empty) strong clique $h^c(e, v)$ and a (possibly empty) strongly stable set $h^s(e, v)$;
(8) for all $e \in E_H$ and $v \in V_H$, if e is incident with v then $h^c(e, v)$ and $h^s(e, v)$ are both nonempty, and if e is not incident with v then $h^c(e, v) = h^s(e, v) = \emptyset$;
(9) $E_1 \cup E_2 = L \cup \bigcup_{e \in E_H} h(e) \cup \bigcup_{e \in E_H} \bigcup_{v \in V_H} h^s(e, v)$;
(10) $E_1 \cap E_2 = \emptyset$;
(11) for all $x \in L$ and $i \in \{1, 2\}$, if $x \in E'_i$ then $x \in E_i$;
(12) for all $e \in E_H$ and $i \in \{1, 2\}$, if $e \in E'_i$ then $h(e) \subseteq E_i$;
(13) for all $e \in E_H$, $v \in V_H$, and all distinct $i, j \in \{1, 2\}$, if $e \in E'_i$ then $h^s(e, v) \subseteq E_j$;
(14) $H'[L] = G[L]$;
(15) for all $x \in L$ and $e \in E_H$, if xe is a strongly adjacent pair in H' then x is strongly complete to $h(e)$, and if xe is a strongly antiadjacent pair in H' then x is strongly anticomplete to $h(e)$;
(16) for all distinct $e, f \in E_H$, if ef is a strongly adjacent pair in H' then $h(e)$ is strongly complete to $h(f)$, and if ef is a strongly antiadjacent pair in H' then $h(e)$ is strongly anticomplete to $h(f)$;
(17) for all $v \in V_H$, if S_v is the set of all vertices in L that have a neighbor in $h(v)$, and T_v is the set of all vertices in $(L \cup \bigcup_{e \in E_H} h(e)) \setminus S_v$ that have a neighbor in S_v, then either $h(v) = \emptyset$ (in which case $S_v = T_v = \emptyset$) or $G[h(v) \cup S_v \cup T_v]$ is a nondegenerate $(h(v), S_v \cap E_1, S_v \cap E_2, T_v \cap E_2, T_v \cap E_1)$-clique connector;
(18) for all distinct $e, f \in E_H$ and $v \in V_H$, $h(e, v)$ is strongly complete to $h(f, v)$;
(19) for all (not necessarily distinct) $e, f \in E_H$ and distinct $u, v \in V_H$, $h(e, u)$ is strongly anticomplete to $h(f, v)$;
(20) for all $e \in E_H$ and $v \in V_H$, $h(e, v)$ is strongly complete to $h(v)$;
(21) for all $e \in E_H$ and distinct $u, v \in V_H$, $h(e, v)$ is strongly anticomplete to $h(u)$;
(22) for all distinct $e, f \in E_H$ and $v \in V_H$, $h(e, v)$ is strongly anticomplete to $h(f)$;
(23) for all $e \in E_H$ with (distinct) endpoints $u, v \in V_H$, $G[h(v) \cup h(e, v) \cup h(e, u)]$ is an $h(e)$-melt such that $h(e) = A$, $h^c(e, v) = K$, $h^c(e, u) = M$, and $h^s(e, v) \cup h^s(e, u) = B$, with $h^s(e, v) = \bigcup_{i=1}^{m} B_{i,0}$ and $h^s(e, u) = \bigcup_{j=1}^{n} B_{0,j}$, where K, M, A, B, m, and n are as in the definition of an A-melt;
(24) for all $e \in E_H$ with (distinct) endpoints $u, v \in V_H$, and all distinct $i, j \in \{1, 2\}$ such that $e \in E'_i$, all of the following hold:
   - $h(e)$ is strongly complete to $(S_u \cup S_v) \cap E_j$,
   - $h(e, v)$ is strongly complete to $S_v \cap E_i$ and strongly anticomplete to $L \setminus (S_v \cap E_i)$,
   - every vertex of $(L \cup \bigcup_{f \in E_H \setminus \{e\}} h(f)) \setminus ((S_u \cup S_v) \cap E_i)$ with a neighbor in $(S_u \cup S_v) \cap E_i$ is strongly complete to $h(e)$.

We leave it to the reader to check that if (H, H', E'_1, E'_2) is a usable 4-tuple and (G, E_1, E_2) admits an (H, H', E'_1, E'_2)-structure, then H is a usable graph, G admits an H-structure, and E_1 and E_2 are both (possibly empty) strongly stable sets.

We say that a trigraph G belongs to the *class* $\mathcal{T}_1^*$ provided that either G is a double melt, or there exist $E_1, E_2 \subseteq V_G$ and a usable 4-tuple (H, H', E'_1, E'_2) such that (G, E_1, E_2) admits an (H, H', E'_1, E'_2)-structure.

We observe that all bipartite trigraphs are in $\mathcal{T}_1^*$, as are all nondegenerate clique connectors (and therefore all tulips), and all melts. Further, we remind the reader that the class $\mathcal{T}_1$ consists of trigraphs G such that either G is a double melt or there exists a usable graph H such that G admits an H-structure. Thus, the class $\mathcal{T}_1^*$ is a subclass of the class $\mathcal{T}_1$.

Our goal for the remainder of this section is to establish that each trigraph in $\mathcal{T}_1^*$ is a tulip bed and therefore Berge, and that each Berge trigraph in $\mathcal{T}_1$ is in $\mathcal{T}_1^*$.

5.8. *Each trigraph in $\mathcal{T}_1^*$ is a tulip bed. Consequently, each trigraph in $\mathcal{T}_1^*$ is Berge.*

Proof. By 5.6, it suffices to prove the first statement. Let $G \in \mathcal{T}_1^*$. If G is a double melt, then we are done by 5.7. So assume that there exists some usable 4-tuple (H, H', E_1', E_2') and some E_1 and E_2 such that (G, E_1, E_2) admits an (H, H', E_1', E_2')-structure. If H is the empty graph, then G is bipartite and therefore a tulip bed; so assume that H is not empty. We use the notation from the definition of a triple (G, E_1, E_2) that admits an (H, H', E_1', E_2')-structure. Set $F_1 = E_1$ and $F_2 = E_2$. We may assume that the vertex set of H is $\{v_1, \dots, v_s\}$ for some integer $s \geq 1$; then for each $l \in \{1, \dots, s\}$, set $Y_l = h(v_l) \cup \bigcup_{e \in E_H} h^c(e, v_l)$. We observe that $Y_1, \dots, Y_s$ partition $V_G \setminus (F_1 \cup F_2)$ into nonempty strong cliques, pairwise strongly anticomplete to each other. Next, for each $l \in \{1, \dots, s\}$ and $e \in E_H$, let $h^l(e)$ be the set of all vertices in $h(e)$ with a neighbor in $h^c(e, v_l)$; then $S_{v_l} \cup \bigcup_{e \in E_H} h^l(e) \cup \bigcup_{e \in E_H} h^s(e, v_l)$ is the set of all vertices in $F_1 \cup F_2 = E_1 \cup E_2$ with a neighbor in Y_l, and so we set $X_l = S_{v_l} \cup \bigcup_{e \in E_H} h^l(e) \cup \bigcup_{e \in E_H} h^s(e, v_l)$. With this setup, it is easy to see that G is a tulip bed. ■

As with melts (see the comment after 5.7), it is possible to get a slightly stronger result than the one that we stated in 5.8. The reader can check that if (G, E_1, E_2) admits an (H, H', E_1', E_2')-structure for some usable 4-tuple (H, H', E_1', E_2'), and if $Y_1, \dots, Y_s$ and $X_1, \dots, X_s$ are constructed as in the proof above, then for each $l \in \{1, \dots, s\}$, $G[Y_l \cup X_l \cup Z_l]$ is a $(Y_l, X_l \cap E_1, X_l \cap E_2, Z_l \cap E_2, Z_l \cap E_1)$-clique connector, where Z_l is the set of all vertices in $(E_1 \cup E_2) \setminus X_l$ with a neighbor in X_l. But we do not need this stronger result, and so we omit the proof.

It remains to show that every Berge trigraph in $\mathcal{T}_1$ is in $\mathcal{T}_1^*$. We begin with a technical lemma.

5.9. *Let H be a usable graph, and let G be a Berge trigraph that admits an H-structure. Then the set L and the function h from the definition of a trigraph that admits an H-structure can be chosen so that for all isolated vertices $v \in V_H$, $G[h(v) \cup S_v \cup T_v]$ is a nondegenerate $(h(v), A_v, B_v, C_v, D_v)$-clique connector (where $S_v, T_v, A_v, B_v, C_v, D_v$ are as in the definition).*

Proof. Let $L \subseteq V_G$ and $h : V_H \cup E_H \cup (E_H \times V_H) \to 2^{V_G \setminus L}$ satisfy the properties laid out in the definition of a trigraph that admits an H-structure. Since G is Berge, by 5.1 we have that for all isolated vertices $v \in V_H$ such that $G[h(v) \cup S_v \cup T_v]$ is a degenerate $(h(v), A_v, B_v, C_v, D_v)$-clique connector, at least one of C_v and D_v is empty. After possibly relabeling, we may assume that for all isolated vertices $v \in V_H$ such that $G[h(v) \cup S_v \cup T_v]$ is a degenerate $(h(v), A_v, B_v, C_v, D_v)$-clique connector, we have that $D_v = \emptyset$. Now, let V_H^d be set of all isolated vertices in V_H such that $G[h(v) \cup S_v \cup T_v]$ is a degenerate $(h(v), A_v, B_v, C_v, \emptyset)$-clique connector, and for all $v \in V_H^d$, set $B_v = \{b_v\}$. Set $\hat{L} = (L \setminus$

$\bigcup_{v\in V_H^d}\{b_v\}) \cup \bigcup_{v\in V_H^d} h(v)$. Next, we define $\hat{h} : V_H \cup E_H \cup (E_H \times V_H) \to 2^{V_G\setminus\hat{L}}$ to be the map that satisfies all of the following:

- for all $v \in V_H^d$, $\hat{h}(v) = \{b_v\}$;
- for all $v \in V_H \setminus V_H^d$, $\hat{h}(v) = h(v)$;
- for all $e \in E_H$, $\hat{h}(e) = h(e)$;
- for all $e \in E_H$ and $v \in V_H$, $\hat{h}(e, v) = h(e, v)$.

Using 5.2, we easily get that $\hat{L}$ and $\hat{h}$ satisfy the requirements from the statement of the theorem. ■

5.10. *Let H be a usable graph, and let G be a Berge trigraph that admits an H-structure. Then H is a bipartite graph. Furthermore, there exists a bipartite trigraph H' such that for every bipartition (E_1', E_2') of H', (H, H', E_1', E_2') is a usable 4-tuple, and there exist some $E_1, E_2 \subseteq V_G$ such that (G, E_1, E_2) admits an (H, H', E_1', E_2')-structure.*

Proof. Let L and h be chosen as in 5.9. We construct H' as follows. The vertex set of H' is $E_H \cup L$. Set $H'[L] = G[L]$. For all $x \in L$ and $e \in E_H$, we let xe be a strongly adjacent pair in H' if x is strongly complete to $h(e)$ in G, and we let xe be a strongly antiadjacent pair in H' if x is strongly anticomplete to $h(e)$ in G; since for all $x \in L$ and $e \in E_H$, x is either strongly complete or strongly anticomplete to $h(e)$ in G, this completely defines the adjacency between L and E_H in H'. Finally, for all distinct $e, f \in E_H$, we let ef be a strongly adjacent pair in H' if $h(e)$ is strongly complete to $h(f)$, and ef is a strongly antiadjacent pair in H' if $h(e)$ is strongly anticomplete to $h(f)$ in G; since for all distinct $e, f \in E_H$, we have that $h(e)$ is either strongly complete or strongly anticomplete to $h(f)$, this completely defines adjacency in $H'[E_H]$. We observe that if distinct $e, f \in E_H$ share an endpoint, then e and f are adjacent in H'.

Note that H' contains no odd holes and no triangles, for otherwise, we would immediately get an odd hole or a triangle, respectively, in $G[L \cup \bigcup_{e\in E_H} E_H]$, which is impossible. Since every realization of H' contains the line graph of H as a (not necessarily induced) subgraph, this implies that H is bipartite. Let (E_1', E_2') be any bipartition of the bipartite trigraph H'. Clearly, (H, H', E_1', E_2') is a usable 4-tuple.

Next, we set:

$$E_1 = (L \cap E_1') \cup \bigcup_{e\in E_H\cap E_1'} h(e) \cup \bigcup_{(e,v)\in(E_H\cap E_2')\times V_H} h^s(e, v);$$

$$E_2 = (L \cap E_2') \cup \bigcup_{e\in E_H\cap E_2'} h(e) \cup \bigcup_{(e,v)\in(E_H\cap E_1')\times V_H} h^s(e, v).$$

By construction, (E_1, E_2) is a partition of the set

$$L \cup \bigcup_{e\in E_H} h(e) \cup \bigcup_{(e,v)\in E_H\times V_H} h^s(e, v).$$

To show that (G, E_1, E_2) admits an (H, H', E_1', E_2')-structure, it suffices to show that for all $v \in V_H$, either $A_v \cup D_v \subseteq E_1$ and $B_v \cup C_v \subseteq E_2$, or $A_v \cup D_v \subseteq E_2$ and $B_v \cup C_v \subseteq E_1$, for then the result will easily follow from the appropriate definitions (together with the choice of L and h). So fix $v \in V_H$; if $h(v) = \emptyset$, then we are done, and so assume that $h(v) \neq \emptyset$. Then by the definition of a clique connector, there exist some $a \in A_v$ and $b \in B_v$

such that a is strongly complete to B_v and b is strongly complete to A_v. In particular, ab is an adjacent pair, and so $a \in E'_i$ and $b \in E'_j$ for some distinct $i, j \in \{1, 2\}$. Since a is strongly complete to B_v, this implies that $B_v \subseteq E'_j$; and similarly, $A_v \subseteq E'_i$. By the construction of E_1 and E_2 then, we get that $A_v \subseteq E_i$ and $B_v \subseteq E_j$. It remains to show that $C_v \subseteq E_j$ and that $D_v \subseteq E_i$; by symmetry, it suffices to prove the former. By definition, there exist some $L_{C_v} \subseteq L$ and $E_{C_v} \subseteq E_H$ such that $C_v = L_{C_v} \cup \bigcup_{e \in E_{C_v}} h(e)$. Now, in H', each member of $L_{C_v} \cup E_{C_v}$ has a neighbor in $A_v \subseteq E'_i$, and so $L_{C_v} \cup E_{C_v} \subseteq E'_j$. It then easily follows that $C_v \subseteq E_j$. ■

We can now finally prove the main result of this section.

5.11. *$\mathcal{T}_1^*$ is the class of all Berge trigraphs in $\mathcal{T}_1$.*

Proof. It is easy to check that $\mathcal{T}_1^* \subseteq \mathcal{T}_1$. By 5.8, every trigraph in $\mathcal{T}_1^*$ is Berge. Now, suppose that G is a Berge trigraph in $\mathcal{T}_1$. If G is a double melt, then $G \in \mathcal{T}_1^*$ by definition. So suppose that G admits an H-structure for some usable graph H. Then by 5.10, there exist some $E_1, E_2 \subseteq V_G$ and some usable 4-tuple (H, H', E'_1, E'_2) such that (G, E_1, E_2) admits an (H, H', E'_1, E'_2)-structure, and consequently, $G \in \mathcal{T}_1^*$ by the definition of the class $\mathcal{T}_1^*$. ■

6. CLASS $\mathcal{T}_2$

In this section, we give the definition of the class $\mathcal{T}_2$ from [5], and we prove that each trigraph in $\mathcal{T}_2$ is Berge. Prior to giving the definition of the class $\mathcal{T}_2$, we note that, by 5.5 from [5], the class $\mathcal{T}_2$ is self-complementary; we state this result below for future reference.

6.1. *The class $\mathcal{T}_2$ is self-complementary, that is, for all $G \in \mathcal{T}_2$, we have that $\overline{G} \in \mathcal{T}_2$.*

Thus, in order to show that each trigraph in $\mathcal{T}_2$ is Berge, it suffices to show that each trigraph in $\mathcal{T}_2$ is odd hole-free.

Informally, trigraphs in the class $\mathcal{T}_2$ are obtained from some basic "building blocks" (namely, "1-thin trigraphs," "2-thin trigraphs," and bipartite and complement-bipartite trigraphs of a certain kind; we define each of these below) by "composing along doubly dominating semiadjacent pairs" (this operation is also defined below). We will show that if a trigraph G is obtained from two trigraphs that do not contain any odd holes by composing along doubly dominating semiadjacent pairs, then G does not contain any odd holes. We will then show that none of our basic "building blocks" contain an odd hole. This will prove that no trigraph in $\mathcal{T}_2$ contains an odd hole, and therefore (by 6.1) that each trigraph in $\mathcal{T}_2$ is Berge.

We begin with some definitions. We say that a homogeneous pair (A, B) in a trigraph G is *doubly dominating* provided that there exist nonempty sets $C, D \subseteq V_G$ such that $(A, B, C, D, \emptyset, \emptyset)$ is the partition of G associated with (A, B). We say that a semiadjacent pair ab in G is *doubly dominating* provided that $(\{a\}, \{b\})$ is a doubly dominating homogeneous pair in G. Next, let G_1 and G_2 be trigraphs with disjoint vertex sets, and for each $i \in \{1, 2\}$, let a_ib_i be a doubly dominating semiadjacent pair in G_i. For each $i \in \{1, 2\}$, let $(\{a_i\}, \{b_i\}, A_i, B_i, \emptyset, \emptyset)$ be the partition of G associated with $(\{a_i\}, \{b_i\})$. We then say that a trigraph G is obtained from G_1 and G_2 by *composing along* (a_1, b_1, a_2, b_2) provided all of the following hold:

- $V_G = A_1 \cup B_1 \cup A_2 \cup B_2$;
- for each $i \in \{1, 2\}$, $G[A_i \cup B_i] = G_i[A_i \cup B_i]$;
- A_1 is strongly complete to A_2 and strongly anticomplete to B_2;
- B_1 is strongly complete to B_2 and strongly anticomplete to A_2.

6.2. *Let G_1 and G_2 be odd hole-free trigraphs with disjoint vertex sets, and for each $i \in \{1, 2\}$, let $a_i b_i$ be a doubly dominating semiadjacent pair in G_i. Let G be the trigraph obtained by composing G_1 and G_2 along (a_1, b_1, a_2, b_2). Then G is odd hole-free.*

Proof. Suppose otherwise. Let W be the vertex set of an odd hole in G, and let $\hat{G}$ be a realization of G such that W is the vertex set of an odd hole in $\hat{G}$. First, note that $G \setminus A_1$ is obtained by substituting $G_1[B_1]$ for the vertex b_2 in $G_2 \setminus \{a_2\}$. Since neither G_1 nor G_2 contains an odd hole, by 2.2, this means that $G \setminus A_1$ contains no odd hole. Thus, W intersects A_1. In an analogous manner, we get that W intersects B_1, A_2, and B_2 as well. Next, since A_1 is complete to A_2 in $\hat{G}$, and since $\hat{G}[W]$ is a chordless cycle of length at least five and therefore contains no vertices of degree greater than two and no (not necessarily induced) cycles of length 4, we know that $|W \cap (A_1 \cup A_2)| \leq 3$; similarly, $|W \cap (B_1 \cup B_2)| \leq 3$. Since $|W|$ is odd, and since W intersects each of A_1, B_1, A_2, and B_2, this means that we may assume by symmetry that $|W \cap A_1| = 2$ and $|W \cap B_1| = |W \cap A_2| = |W \cap B_2| = 1$. Set $W \cap A_1 = \{\hat{a}_1, \hat{a}'_1\}$, $W \cap B_1 = \{\hat{b}_1\}$, $W \cap A_2 = \{\hat{a}_2\}$, and $W \cap B_2 = \{\hat{b}_2\}$. Since $\hat{a}_2$ is complete to $\{\hat{a}_1, \hat{a}'_1\}$ in $\hat{G}$, we know that $\hat{a}_2$ is nonadjacent to $\hat{b}_2$ in $\hat{G}$. But then the only neighbor of $\hat{b}_2$ in $\hat{G}[W]$ is $\hat{a}_2$, which is impossible since $\hat{G}[W]$ is a cycle. ■

We note that in [9], Cornuéjols and Cunningham proved a result similar to 6.2. They showed that a graph operation whose special case is very similar to our operation of composing along doubly dominating semiadjacent pairs preserves perfection.

Triangle-patterns and triad-patterns. Given a graph H, we say that a trigraph G is an *H-pattern* provided that V_G can be partitioned into sets $\{a_v \mid v \in V_H\}$ and $\{b_v \mid v \in V_H\}$ such that all of the following hold:

- $a_v b_v$ is a semiadjacent pair for all $v \in V_H$;
- if $u, v \in V_H$ are adjacent, then $a_u a_v$ and $b_u b_v$ are strongly adjacent pairs, and $a_u b_v$ and $a_v b_u$ are strongly antiadjacent pairs;
- if $u, v \in V_H$ are nondjacent, then $a_u a_v$ and $b_u b_v$ are strongly antiadjacent pairs, and $a_u b_v$ and $a_v b_u$ are strongly adjacent pairs.

We observe that $a_v b_v$ is a doubly dominating semiadjacent pair in G for all $v \in V_H$. Let K_n denote the complete graph on n vertices. We say that a trigraph G is a *triangle-pattern* provided that G is a K_3-pattern, we say that G is a *triad-pattern* provided that G is a $\overline{K}_3$-pattern.

We note that triangle-patterns are complement-bipartite and triad-patterns are bipartite; thus, triangle-patterns and triad-patterns are Berge.

1-thin trigraphs. Let G be a trigraph, and let $a, b \in V_G$ be distinct. Let $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_m\}$ be disjoint, nonempty subsets of $V_G \setminus \{a, b\}$ such that $V_G \setminus \{a, b\} = A \cup B$. Assume that all of the following hold:

(1) ab is a semiadjacent pair;

(2) a is strongly complete to A and strongly anticomplete to B;
(3) b is strongly complete to B and strongly anticomplete to A;
(4) for all $i, j \in \{1, \dots, n\}$ such that $i < j$, if $a_i a_j$ is an adjacent pair, then a_i is strongly complete to $\{a_{i+1}, \dots, a_{j-1}\}$, and a_j is strongly complete to $\{a_1, \dots, a_{i-1}\}$;
(5) for all $i, j \in \{1, \dots, m\}$ such that $i < j$, if $b_i b_j$ is an adjacent pair, then b_i is strongly complete to $\{b_{i+1}, \dots, b_{j-1}\}$, and b_j is strongly complete to $\{b_1, \dots, b_{i-1}\}$;
(6) for all $p \in \{1, \dots, n\}$ and $q \in \{1, \dots, m\}$, if $a_p b_q$ is an adjacent pair, then a_p is strongly complete to $\{b_{q+1}, \dots, b_m\}$, and b_q is strongly complete to $\{a_{p+1}, \dots, a_n\}$.

We then say that G is 1-*thin with base* (a, b), or simply that G is 1-*thin*.

Note that if a trigraph G is 1-thin with base (a, b), then ab is a doubly dominating semiadjacent pair in G. Note also that a trigraph G is 1-thin with base (a, b) if and only if G is 1-thin with base (b, a). Further, the complement of a 1-thin trigraph with base (a, b) is again 1-thin with base (b, a) (or equivalently, with base (a, b)). Indeed if G is 1-thin, then setting $\overline{a} = b$, $\overline{b} = a$, $\overline{a}_i = a_{n-i+1}$ for all $i \in \{1, \dots, n\}$, and $\overline{b}_i = b_{m-i+1}$ for all $i \in \{1, \dots, m\}$, we immediately get that $\overline{G}$ is 1-thin with base $(\overline{a}, \overline{b})$, that is, with base (b, a).

6.3. *Let G be a* 1-*thin trigraph. Then G is odd hole-free.*

Proof. Let $a, b, A = \{a_1, \dots, a_n\}$, and $B = \{b_1, \dots, b_m\}$ be as in the definition of a 1-thin trigraph. We begin by showing that (A, B) is a good homogeneous pair in G. This means that we need to prove the following:

- neither $G[A]$ nor $G[B]$ contains a three-edge path;
- there does not exist a path $v_1 - v_2 - v_3 - v_4$ in G such that $v_1, v_4 \in A$ and v_2, $v_3 \in B$;
- there does not exist a path $v_1 - v_2 - v_3 - v_4$ in G such that $v_1, v_4 \in B$ and v_2, $v_3 \in A$.

We begin by proving the second claim. Suppose that $i, l \in \{1, \dots, m\}$ and $j, k \in \{1, \dots, n\}$ are such that $a_i - b_j - b_k - a_l$ is a three-edge path in G. Since $a_i b_j$ is an adjacent pair, while $a_i b_k$ is an antiadjacent pair, we know that $k < j$. But then since $a_l b_k$ is an adjacent pair, $a_l b_j$ must be a strongly adjacent pair, which is a contradiction. An analogous argument establishes that the third claim holds as well.

Before tackling the first claim, we establish an auxiliary result. Let $i, j, k \in \{1, \dots, n\}$ be such that $a_i - a_j - a_k$ is a path in $G[A]$; we claim that $j < \min\{i, k\}$. Suppose otherwise. By symmetry, we may assume that $i < j$. Then if $k < i < j$, the fact that $a_j a_k$ is an adjacent pair implies that $a_i a_k$ is a strongly adjacent pair; if $i < k < j$, then the fact that $a_i a_j$ is an adjacent pair implies that $a_i a_k$ is a strongly adjacent pair; and if $i < j < k$, then the fact that $a_j a_k$ is an adjacent pair implies that $a_i a_k$ is a strongly adjacent pair. But since $a_i - a_j - a_k$ is a path, $a_i a_k$ is an antiadjacent pair, which is a contradiction.

Now, suppose that $i, j, k, l \in \{1, \dots, n\}$ are such that $a_i - a_j - a_k - a_l$ is a path in G. Then since $a_i - a_j - a_k$ is a path in $G[A]$, we have by the above that $j < \min\{i, k\}$; similarly, since $a_j - a_k - a_l$ is a path in $G[A]$, we have that $k < \min\{j, l\}$. But it then follows that $j < k$ and that $k < j$, which is impossible. Thus, $G[A]$ contains no three-edge paths. We get in an analogous fashion that $G[B]$ contains no three-edge paths, and so (A, B) is a good homogeneous pair.

Now, suppose that G contains an odd hole, and let W be the vertex set of an odd hole in G. By 2.3, we get that $|W \cap A| \le 1$ and $|W \cap B| \le 1$. But since $V_G = \{a, b\} \cup A \cup B$, this means that $|W| \leq 4$, which is impossible since an odd hole must have at least five vertices. ■

2-thin trigraphs. Let G be a trigraph. Let $V_G = A \cup B \cup K \cup M \cup \{x_{AK}, x_{AM}, x_{BK}, x_{BM}\}$, where $x_{AK}, x_{AM}, x_{BK}, x_{BM}$ are pairwise distinct vertices, and A, B, K, M, and $\{x_{AK}, x_{AM}, x_{BK}, x_{BM}\}$ are pairwise disjoint. Let $t, s \geq 0$, and let $K = \{k_1, \ldots, k_t\}$ and $M = \{m_1, \ldots, m_s\}$ (so if $t = 0$ then $K = \emptyset$, and if $s = 0$ then $M = \emptyset$). Let A be the disjoint union of sets $A_{i,j}$ and let B be the disjoint union of the sets $B_{i,j}$, where $i \in \{0, \ldots, t\}$ and $j \in \{0, \ldots, s\}$. Assume that:

(1) A and B are (possibly empty) strongly stable sets;
(2) K and M are (possibly empty) strong cliques;
(3) A is strongly complete to B;
(4) K is strongly anticomplete to M;
(5) A is strongly complete to $\{x_{AK}, x_{AM}\}$ and strongly anticomplete to $\{x_{BK}, x_{BM}\}$;
(6) B is strongly complete to $\{x_{BK}, x_{BM}\}$ and strongly anticomplete to $\{x_{AK}, x_{AM}\}$;
(7) K is strongly complete to $\{x_{AK}, x_{BK}\}$ and strongly anticomplete to $\{x_{AM}, x_{BM}\}$;
(8) M is strongly complete to $\{x_{AM}, x_{BM}\}$ and strongly anticomplete to $\{x_{AK}, x_{BK}\}$;
(9) $x_{AK}x_{BM}$ and $x_{AM}x_{BK}$ are semiadjacent pairs;
(10) $x_{AK}x_{BK}$ and $x_{AM}x_{BM}$ are strongly adjacent pairs;
(11) $x_{AK}x_{AM}$ and $x_{BK}x_{BM}$ are strongly antiadjacent pairs;
(12) for all $i, i' \in \{0, \ldots, t\}$ and $j, j' \in \{0, \ldots, s\}$, if $i < i'$ and $j < j'$, then at least one of the sets $A_{i,j}$ and $A_{i',j'}$ is empty, and at least one of the sets $B_{i,j}$ and $B_{i',j'}$ is empty;
(13) for all $i \in \{0, \ldots, t\}$ and $j \in \{0, \ldots, s\}$, all of the following hold:
  – $A_{i,j}$ is strongly complete to $\{k_1, \ldots, k_{i-1}\} \cup \{m_{s-j+2}, \ldots, m_s\}$,
  – $A_{i,j}$ is complete to $\{k_i, m_{s-j+1}\}$,
  - $A_{i,j}$ is strongly anticomplete to $\{k_{i+1}, \ldots, k_t\} \cup \{m_1, \ldots, m_{s-j}\}$,
  - $B_{i,j}$ is strongly complete to $\{k_{t-i+2}, \ldots, k_t\} \cup \{m_1, \ldots, m_{j-1}\}$,
  – $B_{i,j}$ is complete to $\{k_{t-i+1}, m_j\}$,
  – $B_{i,j}$ is strongly anticomplete to $\{k_1, \ldots, k_{t-i}\} \cup \{m_{j+1}, \ldots, m_s\}$.

Then we say that G is 2-*thin with base* $(x_{AK}, x_{BM}, x_{BK}, x_{AM})$, or simply that G is 2-*thin*. We call (A, B, K, M) the *partition of G with respect to the base* $(x_{AK}, x_{BM}, x_{BK}, x_{AM})$.

Suppose that G is a 2-thin trigraph with base $(x_{AK}, x_{BM}, x_{BK}, x_{AM})$. It is then easy to see that G is 2-thin with base $(x_{BK}, x_{AM}, x_{AK}, x_{BM})$; it is also easy to see that $\overline{G}$ is 2-thin with base $(x_{AK}, x_{BM}, x_{AM}, x_{BK})$. Next, note $x_{AK}x_{BM}$ and $x_{BK}x_{AM}$ are both doubly dominating semiadjacent pairs, and G contains no other doubly dominating semiadjacent pairs. We also observe that G is 1-thin with base (x_{AK}, x_{BM}), and also with base (x_{BK}, x_{AM}). By 6.3 then, 2-thin trigraphs are odd hole-free.

The class $\mathcal{T}_2$. Let $k \geq 1$ be an integer, and let $G'_1, \ldots, G'_k$ be trigraphs, such that for all $i \in \{1, \ldots, k\}$, G'_i is either a triangle-pattern or a triad-pattern or a 2-thin trigraph. For each $i \in \{2, \ldots, k\}$, let $c_i d_i$ be a doubly dominating semiadjacent pair in G'_i. For each $j \in \{1, \ldots, k-1\}$, let $x_j y_j$ be a doubly dominating semiadjacent pair in G'_q for some $q \in \{1, \ldots, j\}$. Assume that $\{c_2, d_2\}, \ldots, \{c_k, d_k\}, \{x_1, y_1\}, \ldots, \{x_{k-1}, y_{k-1}\}$ are pairwise distinct (and therefore pairwise disjoint). Let $G_1 = G'_1$, and for each $i \in \{1, \ldots, k-1\}$, let G_{i+1} be the trigraph obtained by composing G_i and G'_{i+1} along $(x_i, y_i, c_{i+1}, d_{i+1})$. Let $G = G_k$. We call such a trigraph G a *skeleton*. We observe that a semiadjacent pair uv in G is doubly dominating in G if and only if uv is a doubly dominating semiadjacent pair in G'_i for some $i \in \{1, \dots, k\}$ and $\{u, v\}$ is not among $\{c_2, d_2\}, \dots, \{c_k, d_k\}, \{x_1, y_1\}, \dots, \{x_{k-1}, y_{k-1}\}$.

The class $\mathcal{T}_2$ consists of all skeletons, and of all trigraphs G that can be obtained as follows. Let G'_0 be a skeleton, and let $n \geq 1$ be an integer. Let $a_1b_1, \dots, a_nb_n$ be doubly dominating semiadjacent pairs in G'_0 such that $\{a_1, b_1\}, \dots, \{a_n, b_n\}$ are pairwise distinct (and therefore pairwise disjoint). For each $i \in \{1, \dots, n\}$, let G'_i be a trigraph such that:

(1) $V_{G'_i} = A_i \cup B_i \cup \{a'_i, b'_i\}$;
(2) the sets $A_i, B_i, \{a'_i, b'_i\}$ are all nonempty and pairwise disjoint;
(3) a'_i is strongly complete to A_i and strongly anticomplete to B_i;
(4) b'_i is strongly complete to B_i and strongly anticomplete to A_i;
(5) a'_i is semiadjacent to b'_i, and either
   - both A_i, B_i are strong cliques, and there do not exist $a \in A_i$ and $b \in B_i$, such that a is strongly anticomplete to $B_i \setminus \{b\}$, b is strongly anticomplete to $A_i \setminus \{a\}$, and a is semiadjacent to b, or
   - both A_i, B_i are strongly stable sets, and there do not exist $a \in A_i$ and $b \in B_i$, such that a is strongly complete to $B_i \setminus \{b\}$, b is strongly complete to $A_i \setminus \{a\}$, and a is semiadjacent to b, or
   - G'_i is a 1-thin trigraph with base (a'_i, b'_i), and G'_i is not a 2-thin trigraph.

We observe that for all $i \in \{1, \dots, n\}$, $a'_ib'_i$ is a doubly dominating semiadjacent pair in G'_i, and if uv is a doubly dominating semiadjacent pair in G'_i, then $\{u, v\} = \{a'_i, b'_i\}$. Now, let $G_0 = G'_0$, and for $i \in \{1, \dots, n\}$, let G_i be obtained by composing G_{i-1} and G'_i along (a_i, b_i, a'_i, b'_i). Let $G = G_n$.

6.4. *Every trigraph in $\mathcal{T}_2$ is Berge.*

Proof. By 6.1, it suffices to show that every trigraph in $\mathcal{T}_2$ is odd hole-free. Recall that 2-thin trigraphs are 1-thin, that triangle-patterns are complement-bipartite, and that triad-patterns are bipartite. Thus, each trigraph in $\mathcal{T}_2$ is obtained by successively composing 1-thin trigraphs, bipartite trigraphs, and complement-bipartite trigraphs along doubly dominating semiadjacent pairs. Since bipartite and complement-bipartite trigraphs are odd hole-free, the result follows from 6.2 and 6.3 . ■

We end this section by stating a few results from [4] that help us understand the structure of trigraphs in the class $\mathcal{T}_2$. By 6.7 from [4], all 1-thin trigraphs (and therefore, all 2-thin trigraphs) are in $\mathcal{T}_2$. By 6.6 from [4], all bipartite trigraphs with a doubly dominating semiadjacent pair are in $\mathcal{T}_2$; and since $\mathcal{T}_2$ is closed under complementation, it follows that all complement-bipartite trigraphs with a doubly dominating semiadjacent pair are in $\mathcal{T}_2$. Finally, by 6.8 from [4], the class $\mathcal{T}_2$ is closed under composing along doubly dominating semiadjacent pairs.

7. THE MAIN THEOREM

In this section, we restate and prove 3.4, the structure theorem for bull-free Berge trigraphs.

3.4. Let G be a trigraph. Then G is a bull-free Berge trigraph if and only if at least one of the following holds:

- G is obtained from smaller bull-free Berge trigraphs by substitution;
- G or $\overline{G}$ is an elementary expansion of a trigraph in $\mathcal{T}_1^*$;
- G is an elementary expansion of a trigraph in $\mathcal{T}_2$. ■

Proof. We first prove the "if" part. If G is obtained by substitution from smaller bull-free Berge trigraphs, then G is bull-free and Berge by 2.2. Next, suppose that G or $\overline{G}$ is an elementary expansion of a trigraph in $\mathcal{T}_1^*$; since G is bull-free and Berge if and only if $\overline{G}$ is, we may assume that G is an elementary expansion of a trigraph in $\mathcal{T}_1^*$. Since $\mathcal{T}_1^*$ is a subclass of $\mathcal{T}_1$, G is an elementary expansion of a trigraph in $\mathcal{T}_1$, and so G is bull-free by 3.3; G is Berge by 4.2 and 5.11. Finally, suppose that G is an elementary expansion of a trigraph in $\mathcal{T}_2$. Then G is bull-free by 3.3 and Berge by 4.2 and 6.4. This proves the "if" part.

To prove the "only if" part, suppose that G is a bull-free Berge trigraph. If G contains a proper homogeneous set, then G is obtained by substitution from smaller bull-free Berge trigraphs, and we are done. So assume that G contains no proper homogeneous sets. Then by 3.2 and 3.3, one of the following holds:

- G or $\overline{G}$ is an elementary expansion of a trigraph in $\mathcal{T}_1$;
- G is an elementary expansion of a trigraph in $\mathcal{T}_2$.

If the latter outcome holds, then we are done. So assume that G or $\overline{G}$ is an elementary expansion of a trigraph $H \in \mathcal{T}_1$. Since G (and therefore $\overline{G}$ as well) is Berge, by 4.2, H is Berge. By 5.11 then, $H \in \mathcal{T}_1^*$. This completes the argument. ■

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