On Building a Knowledge Base for Stability Theory

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Abstract. A lot of mathematical knowledge has been formalized and stored in repositories by now: Different mathematical theorems and theories have been taken into consideration and included in mathematical repositories. Applications more distant from pure mathematics, however — though based on these theories — often need more detailed knowledge about the underlying theories. In this paper we present an example Mizar formalization from the area of electrical engineering focusing on stability theory which is based on complex analysis. We discuss what kind of special knowledge is necessary and which amount of this knowledge is included in existing repositories.

1 Introduction

The aim of mathematical knowledge management is to provide both tools and infrastructure supporting the organization, development, and teaching of mathematics with the help of effective up-to-date computer technologies. To achieve this ambitious goal it should be taken into account that the predominant part of potential users will not be professional mathematicians themselves, but rather scientists or teachers that apply mathematics in their special domain. To attract these people it is essential that our repositories provide a sufficient knowledge base for those domains. We are interested in how far existing mathematical repositories are from meeting this precondition, or in other words: How large is the gap between the knowledge already included in repositories and the knowledge necessary for particular applications?

This problem, however, concerns not only the simple question how much knowledge of a domain is available in a repository. We believe, that in order to measure this gap, it is equally important to consider the basic conditions for a successful formalization of applications on top of existing knowledge, that is on top of a mathematical repository: The more easy such a formalization is, the more attractive is a mathematical repository. To describe attractiveness of a repository for an application one can identify three major points:

1. Amount of knowledge
   This is the obvious question of how much knowledge of a particular domain already has been formalized and included in the repository. Basically, the more knowledge of a domain is included the more attractive a repository is for applications.

2. Representation of knowledge
   This concerns the question of how the knowledge has been defined and formalized: Often mathematicians use more abstract constructions than necessary — and attractive — for applications. An example is the construction of rational functions from polynomials.

3. Applicability of knowledge
   This point deals with both how the knowledge of a domain is organized in a repository and the question of how easy it is to adapt available knowledge to one's own purposes.

In this paper we focus on electrical engineering, in particular on network stability [Unb93]. Network theory deals with the mathematical description, analysis and synthesis of electrical (e.g. continuous, time-discrete or digital) networks. For a reliable application such systems have to be (input/output-) stable, that is for an arbitrary bounded input the output have to be bounded again. In practice it is impossible to verify responses for all input signals. In this situation there is, however, a number of theorems permitting easier methods to decide whether a network is stable [Unb93]. We shall introduce the mathematical fundamentals and prequisites of one example theorem and present a Mizar formalization of this theorem. After that we discus our formalization in the spirit of the three points from above: How far are we from providing a suitable mathematical repository for applications in stability theory?

2 Stability of Networks

As mentioned in the introduction the (input/output-) stability of networks is one of the main issues when dealing with the analysis and design of electrical circuits and systems. In the following we briefly review definitions and properties of electrical systems necessary to understand the rest of the paper. In electrical engineering stability applies to the input/output behaviour of networks (see figure 1). For (time-) continuous systems one finds the following definition. For discrete systems an analogous definition is used.

Definition 1. ([Unb93])
A continuous system is (BIBO-)[3] stable, if and only if each bounded input signal $x(t)$ results in a bounded output signal $y(t)$.

[3] BIBO stands for Bounded Input Bounded Output.

Physically realizable, linear time-invariant systems (LTI systems) can be described by a set of differential equations [Unb93]. The behaviour of a LTI system then is completely characterized by its impulse response $h(t)$.[4] If the impulse response of such a system is known, the relation between the input $x(t)$ and the output $y(t)$ is given by the convolution integral

$$y(t) \;=\; \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau. \tag{1}$$

Furthermore, a LTI system is stable, if and only if its impulse response $h(t)$ is absolute integrable, that is there exists a constant K such that

$$\int_{-\infty}^{\infty} |h(\tau)|\,\mathrm{d}\tau \;\leq\; K \;<\; \infty. \tag{2}$$

In network and filter analysis and design, however, one commonly employs the frequency domain rather than the time domain. To this end the system is described based on its transfer function $H(s)$. In case the Laplace transformation is used we have[5]

$$H(s) \;=\; \int_{-\infty}^{\infty} h(t)e^{-st}\mathrm{d}t. \tag{3}$$

where $s = \sigma + j\omega$ is a complex variable with $\Re\{s\} = \sigma$ and $\Im\{s\} = \omega$.

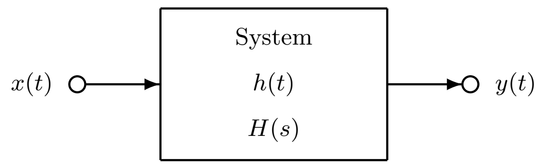


Figure 1: LTI system with one input $x(t)$ and one output $y(t)$

The evaluation of $H(s)$ for $s = j\omega$ — in case of convergence[6] — enables the qualitative understanding of how the system handles and selects various frequencies ω, so for example whether the system describes a high-pass filter, low-pass filter, etc. Now the necessary condition to demonstrate the stability of LTI systems in the frequency domain reduces to show, that the $j\omega$-axis lies in the Laplace transformation's region of convergence (ROC).

[4] $h(t)$ is the output of the system, when the input is the Dirac delta function $\delta(t)$.
[5] Note that this is a generalization of the continuous-time Fourier transformation.
[6] In this case $H(j\omega)$ equals the Fourier transform.

For physically realizable LTI systems, such as the class of networks with constant and concentrated parameters (so-called lumped parameters circuits), $H(s)$ is given in form of a rational function with real coefficients, that is

$$H(s) \ = \ \frac{a_n s^n + \ldots + a_0}{b_m s^m + \ldots + b_0}, \quad a_i, b_i \in \mathbb{R}. \tag{4}$$

In this case the region of convergence can be described by the roots of the denominator polynomial: If $s_i = \sigma_i + j\omega_i$ for $i = 1, \ldots m$ are the roots of $b_m s^m + \ldots + b_0$, the region of convergence is given by

$$\Re\{s\} \ > \max\{\sigma_i, \ i = 1, \ldots m\}.$$

To check stability it is therefore sufficient, to show that the real part $\Re\{s\}$ of all poles of $H(s)$ is smaller then 0. The denominator of $H(s)$ is thus a so-called Hurwitz polynomial.

Note that the stability problem for discrete-time signals and systems can be analized with the same approach. For a given discrete-time transfer function $H(z)$ in the Z- domain, it has to be checked whether the unit circle is contained in the region of convergence. Hence for all poles z_i of $H(z)$ we must have $|z_i| < 1$. Using bilinear transformations [OS98]

$$z \ := \ \frac{1+s}{1-s}. \tag{5}$$

it is thus sufficient to check whether the denominator of

$$H(z)|_{z:=\frac{1+s}{1-s}} \tag{6}$$

is a Hurwitz polynomial.

The practical proof of stability of high-precision filters, however, turns out to be very hard. In practical applications the poles of concern are usually close to the axis $s = j\omega$ or the unit circle $|z| = e^{j\omega}$ respectively. Thus numerical determination of the poles is highly error-prone due to its rounding effects. In digital signal processing in addition degrees of transfer functions tend to be very high, for example 128 and higher in communication networks.

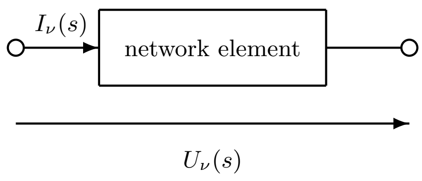
An interesting and in practice often used method to check the stability of a given network is based on the following theorem.

Theorem 1. ([Unb93])
Let $f(x)$ be a real polynomial with degree $n \geq 1$. Furthermore let all coefficients of $f(x)$ be greater than 0. Let $f_e(x)$ and $f_o(x)$ denote the even part resp. the odd part of $f(x)$. Assume further that

$$Z(x) \ = \ \frac{f_e(x)}{f_o(x)}$$

or the reciprocal of $Z(x)$ is a reactance one-port function of degree n. Then $f(x)$ is a Hurwitz polynomial.

The concept of reactance one-port function stems from electrical network theory: In arbitrary passive (that is RLC-) networks we find the following relations between the complex voltage $U_\nu(s)$ and the complex current $I_\nu(s)$:



$$\begin{aligned}
U_r(s) &= R_r \cdot I_r(s) && \text{for a resistor } R_r \\
U_l(s) &= s \cdot L_l \cdot I_l(s) && \text{for an inductance } L_l \\
U_k(s) &= \tfrac{1}{s \cdot C_k} \cdot I_k(s) && \text{for a capacity } C_k
\end{aligned}$$

An impedance (complex resistor) or admittance (complex conductance) composed of network elements R, L and C only is called a (RLC-) one-port function, an impedance or admittance composed of network elements L and C only is called a reactance one-port function. Conversely, for every one-port function $Z(s)$ there exists at least one one-port, whose impedance or admittance is equal to $Z(s)$:

Hence, theorem 1 reduces stability checking to the considerable easier task to synthesize a one-port solely using inductors (L) and capacitors (C), that is to synthesize a reactance one-port. To this end there exist easy procedures like for example Routh's method to construct a chain one-port [Unb93].

It turns out that one-port functions $Z(s)$ are exactly the real positive rational functions. For a real function we have that for real s also $Z(s)$ is real[7], and a positive function means that $\Re\{s\} > 0$ implies $\Re\{Z(s)\} > 0$. A reactance one-port function is a one-port function, that is in addition odd. The property of being positive is closely connected to Hurwitz polynomials:

Theorem 2. ([Unb93])
Let $f(x)$ be a real polynomial with degree $n \geq 1$. Furthermore let all coefficients of $f(x)$ be greater than 0. Let $f_e(x)$ and $f_o(x)$ denote the even part resp. the odd part of $f(x)$. Assume that$f_e(x)$ and $f_o(x)$ have no common roots and that $Z(x) = f_e(x)/f_o(x)$ is positive. Then

(i) $\Re\{Z(x)\} \geq 0$ for all x with $\Re\{x\} = 0$
(ii) $f_e(x) + f_o(x)$ is a Hurwitz polynomial.

[7] This condition implies that the coefficients of $Z(s)$ are real. In network theory, however, this definition is used.

In section 4.2 we will see that this theorem is also the key to prove that stability checking can be reduced to synthesizing LC-one-ports, in other words to prove theorem 1 from above.

3 The Mizar System

The logical basis of Mizar [RT01,Miz10] is classical first order logic extended, however, with so-called schemes. Schemes introduce free second order variables, in this way enabling among others the definition of induction schemes. In addition Mizar objects are typed, the types forming a hierarchy with the fundamental type `set`. The user can introduce new (sub)types describing mathematical objects such as groups, fields, vector spaces or polynomials over rings or fields. To this end the Mizar language provides a powerful typing mechanism based on adjective subtypes [Ban03].

The current development of Mizar relies on Tarski-Grothendieck set theory — a variant of Zermelo Fraenkel set theory using Tarski's axiom on arbitrarily large, strongly inaccessible cardinals [Tar39] which can be used to prove the axiom of choice —, though in principle the Mizar language can be used with other axiom systems also. Mizar proofs are written in natural deduction style as presented in the calculus of [Jaś34]. The rules of the calculus are connected with corresponding (English) natural language phrases so that the Mizar language is close to the one used in mathematical textbooks. The Mizar proof checker verifies the individual proof steps using the notion of obvious inferences [Dav81] to shorten the rather long proofs of pure natural deduction.

4 Mizar Formalization of the Theorem

4.1 Preliminaries About Rational Functions

Although the theory of polynomials in Mizar is rather well developed, rational functions have not been defined yet. In the following we briefly review how we have introduced rational functions in Mizar. Rational functions can — analogously to polynomials — be defined over arbitrary fields: Rational functions are simply pairs of polynomials whose second component is not the zero polynomial.[8] These can be easily introduced as a Mizar type `Rational_function of L`, where `L` is the underlying coefficient domain.

```
definition
let L be non trivial multLoopStr_0;
mode rational_function of L means
  ex p1 being Polynomial of L st
  ex p2 being non zero Polynomial of L st it = [p1,p2];
end;
```

[8] Of course rational functions can be introduced "more algebraically" as the quotient field of a polynomial ring. Here we decided to use pairs to concentrate on application issues; see the discussion in section 5.

Note that in Mizar the result type of the pair constructor `[ , ]` and the projections `‘1` and `‘2`, in the original definition is simply `set`. These, however, can be modified into `Rational_function` and `(non zero) Polynomial` respectively by employing `redefinitions`. In addition one can introduce the usual functions `num` and `denom` as synonyms for the corresponding projections.

```
definition
let L be non trivial multLoopStr_0;
let p1 be Polynomial of L;
let p2 be non zero Polynomial of L;
redefine func [p1,p2] -> rational_function of L;
end;

definition
let L be non trivial multLoopStr_0;
let z be rational_function of L;
redefine func z‘1 -> Polynomial of L;
redefine func z‘2 -> non zero Polynomial of L;
end;

notation
let L be non trivial multLoopStr_0;
let z be rational_function of L;
synonym num(z) for z‘1;
synonym denom(z) for z‘2;
end;
```

Now that `num` and `denom` — applied to rational functions — have result type `Polynomial`, operations for rational functions can straightforwardly be defined by employing the corresponding functions for polynomials. So, for example, the evaluation of rational functions can be defined using evaluation of polynomials and the division operator `/` defined for arbitrary fields `L`.

```
definition
let L be Field;
let z be rational_function of L;
let x be Element of L;
func eval(z,x) -> Element of L equals
  eval(num(z),x) / eval(denom(z),x);
end;
```

Note that according to the definition of `eval` for polynomials the type of the first argument — that is of `num(z)` and `denom(z)` — has to be `Polynomial`. This is ensured by the redefinitions from above, which in this sense allow for reusing the operations defined for polynomials in the case of rational functions. Other necessary operations for rational functions such as the degree or arithmetic operations can be defined the same way.

4.2 The Theorem

To apply the general Mizar theory of polynomials and rational functions with complex numbers we just instantiate the parameter `L` describing the coefficient domain with the field of complex numbers `F_Complex` from [Mil01a]. So an object of type

```
rational_function of F_Complex
```

combines the theory of rational functions with the one of complex numbers.

Further properties necessary to state the main theorem are introduced by defining appropriate attributes for polynomials and rational functions resp. Note that these definitions apply to polynomials and rational functions over the complex numbers only.

```
definition
let p be Polynomial of F_Complex;
attr p is real means
  for i being Element of NAT holds p.i is real number;
end;

definition
let p be rational_function of F_Complex;
attr Z is positive means
  for x being Element of F_Complex
  holds Re(x) > 0 implies Re(eval(Z,x)) > 0;
end;
```

Using these attributes — and the attribute `odd` describing odd functions — we can then introduce one-ports and reactance one-ports in Mizar by the following mode definitions.

```
definition
mode one_port_function is real positive rational_function of F_Complex;

mode reactance_one_port_function is
                      odd real positive rational_function of F_Complex;
end;
```

We also needed to define the odd and the even part of a polynomial `f`. This is accomplished by two Mizar functors `even_part(f)` and `odd_part(f)`, which however can be defined straightforwardly. Finally we formalize the condition from the theorem, that all the coefficients of the given polynomial f should be greater than 0 as usual as a Mizar attribute:

```
definition
let f be real Polynomial of F_Complex;
attr f is with_all_coefficients_positive means
  for i being Element of NAT st i <= deg p holds p.i > 0;
end;
```

Note that for a real polynomial f where all coeffiecients of f are greater than 0 and $\deg(f) \geq 1$ both the even and the odd part of f are not 0, hence both can appear as the denominator of a rational function. Thus prepared we can state theorem 1 from section 2 in Mizar. Note again that due to the redefinitions of section 4.1 the functor `[ , ]` returns a rational function.

```
theorem
for p be non constant with_positive_coefficients
        (real Polynomial of F_Complex)
st [even_part(p),odd_part(p)] is reactance_one_port_function &
   degree([even_part(p),odd_part(p)]) = degree p
holds p is Hurwitz;
```

The proof of the theorem, as already indicated, basically relies on theorem 2 from section 2, which connects rational functions with the property of being a Hurwitz polynomial. Based on our development from above one can formulate this theorem as follows.

```
theorem
for p be non constant with_positive_coefficients
        (real Polynomial of F_Complex)
st [even_part(p),odd_part(p)] is positive &
   even_part(p),odd_part(p) have_no_common_roots
holds (for x being Element of F_Complex
       st Re(x) = 0 & eval(odd_part(p),x) <> 0
       holds Re(eval([even_part(p),odd_part(p)],x)) >= 0) &
      even_part(p) + odd_part(p) is Hurwitz;
```

The corresponding Mizar proof is rather technical. The basic idea consists of considering in addition to

$$Z(x) \;=\; \frac{p_e(x)}{p_o(x)} \tag{7}$$

the rational function

$$W(x) \;=\; \frac{Z(x)-1}{Z(x)+1} = \frac{p_e(x)-p_o(x)}{p_e(x)+p_o(x)}. \tag{8}$$

and to analyze the absolute values $|W(x)|$. If $Z(x)$ is positive, then $|W(x)| \leq 1$ for all x with $\Re\{x\} \geq 0$, which implies that $W(x)$ has no poles for $\Re\{x\} \geq 0$. Thus the denominator polynomial of $W(x)$ can have roots only for $\Re(x) < 0$, so $p_e(x) + p_o(x) = p(x)$ is a Hurwitz polynomial.

The main theorem now easily follows from theorem 2, because the degree condition implies that $p_e(x)$ and $p_o(x)$ have no common roots.

5 Discussion — Lessons Learned

In the following we discuss our formalization from the last section with respect to the three criteria presented in the introduction. Though carried out for Mizar and the Mizar Mathematical Libray (MML) we believe that the situation in other repositories is similar, so that most of our results should hold in a more general context also.

In [RS07] we already presented a Mizar formalization of Schur's theorem, another helpful criterion for stability checking. Based on polynomials only its formalization was rather harmless. The only missing point that caused some work was division of polynomials. However, as we will see, stability checking in general needs definitely more extension than in this case.

5.1 Amount of Knowledge

Complex numbers and polynomials (over arbitrary rings) are included in MML. A lot of theorems have been proved here, so that almost all we needed could be found in the repository. Interestingly rational functions — a rather basic structure — had not been defined, yet. The reason might be that rational functions are mathematically rather simple and this is the first time that a theorem relying on rational functions has been formalized.

Though even and odd functions were already included in MML, the even and odd part of a polynomial was not. This, however, comes with no surprise, just because these polynomial operations are rather seldom used. We hence had to prove a number of theorems dealing with these polynomials, most of them however being elementary like for example

```
theorem
for p being real Polynomial of F_Complex
for x being Element of F_Complex st Re(x) = 0
holds Re(eval(odd_part(p),x)) = 0;
```

Summarizing, except for the lack of rational functions, MML provides the amount of knowledge for stability checking one could expected.

5.2 Representation of Knowledge

The construction of rational functions can be performed in different ways. Of course, one can define rational functions as pairs of polynomials. On the other hand there is the possibility to construct (the field of) rational functions as the completion of polynomial rings. Though the second version is mathematically more challenging we decided to use pairs. We wanted to emphasize the contribution to applications by concentrating on prior knowledge of potential users: Electrical engineers are probably not interested in (working with!) abstract algebra, their interests and needs are different.

In the same context there is another representational problem: In MML we find both the complex numbers and the field of complex numbers. Not a problem in itself, this may cause some confusion when searching for notions and theorems: The functors `Re` and `Im` giving the real and imaginary part, for exmaple, are defined for complex numbers only, thus — theoretically — not applicable to elements of a field. In Mizar, however, this is not necessarily the case: Using a special registration — `identify` [Kor09] — the user can identify terms and operations from different structures, here complex numbers with elements of the field of complex numbers:

```
registration
  let a,b be complex number;
  let x,y be Element of F_Complex;
  identify x+y with a+b when x=a, y=b;
  identify x*y with a*b when x=a, y=b;
end;
```

In effect, after this registration functors `Re` and `Im` are applicable to elements of the field of complex numbers.

In general, different views on mathematical objects — here, complex numbers as numbers or elements of a field — have to be handled carefully in mathematical repositories in order to not confuse possible users. Even the rudimentary difference between a polnomial and its polynomial function can lead to surprises and incomprehension for people not familiar with the formal treatment of mathematics in repositories.

5.3 Applicability of Knowledge

As we have already seen, the adaption of general knowledge in MML to special cases is straightforward: One just instantiates parameters describing the general domain with the special one, so for example `Polynomial of F_Complex` for polynomials over the (field of) complex numbers.

This, on the other hands, means that to work with such instantiations the user has to apply theorems about the general structure. Though highly desirable from the mathematical point of view, it is not clear whether this is really convenient for application users: To work in the special field of complex numbers, for example, then means to search for helpful theorems in the theory of fields, rings or even groups and semigroups. Maybe here a search tool that generates and collects theorems for special instances of theories would be a reasonable help.

The organization of MML is mainly by articles in which authors prove not only their main theorems, but also whatever is necessary and not found in MML. As a consequence theorems of the same topic, e.g. polynomials, can be spread over the repository. A step to overcome this shortcoming is an ongoing project called Mizar encyclopedia building articles with monographic character whose contents is semi-automatically extracted from contributed Mizar articles. Unfortunately polynomials have not been considered in this project, yet.

Summarizing the Mizar system though flexible in order to support special applications lacks an organization of its corresponding repository to support application users in their efforts.

6 Conclusions

We have presented a Mizar formalization of a theorem for stability checking and have discussed how the knowledge contained in MML supported the process from an application user's point of view. Here we want to emphasize two points.

First, when building a knowledge base for an application area, it is difficult to foresee what knowledge is necessary. We have seen that the formalization of Schur's theorem went through without major problems, while the present theorem caused definitely more work and preparation. Furthermore, there are theorems on stability checking using even involved mathematical techniques such as e.g. analytic functions and the maximum principle.

Second, attractiveness of mathematical repositories does not only depend on the amount of knowledge included. Equally important are a clear representation and organization of knowledge so that it in particular stays familiar for users outside the mathematical community.

Consequently is it essential to communicate with experts from the application area. If we want our repositories to be widely used, we have both to provide a reasonable knowledge base and to take care of the fact that application users might represent mathematical knowledge in a different way we are used to. To this end, we need much more formalizations in application domains. Analysis of such formalizations may then lead to methodologies for building applications on top of mathematical repositories and hence may close the gap between knowledge already included in repositories and knowledge necessary for particular applications.

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