Towards Mathematical Knowledge Management for Electrical Engineering

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Abstract. We explore mathematical knowledge in the field of electrical engineering and claim that electrical engineering is a suitable area of application for mathematical knowledge management: We show that mathematical knowledge arising in electrical engineering can be successfully handled by existing MKM systems, namely by the Mizar system. To this end we consider in this paper network theory and in particular stability of networks. As an example for mathematical knowledge in electrical engineering we present a Mizar formalization of Schur's theorem. Schur's theorem provides a recursive, easy method to check for BIBO-stability of networks.

1 Introduction

The aim of mathematical knowledge management is to provide both tools and infrastructure supporting the organization, development, and teaching of mathematics with the help of effective up-to-date computer technologies. To achieve this ambitious goal it should be taken into account that the predominant part of potential users will not be professional mathematicians themselves, but rather scientists or teachers that apply mathematics in their area. This point has been adressed lately with the consideration of physics [HKS06] or geo-sciences [Ses07]. In this paper we inspect another application area for mathematical knowledge management: electrical engineering.

The situation of mathematics in electrical engineering is — as in other engineering sciences — twofold. On the one hand there is a number of areas, such as for example network theory, control engineering or filter design, based on clean mathematical fundamentals and results. On the other hand, however, even in these areas the newest developments often do not rely on these results. Electrical engineers essentially use systems like MathLab or Maple providing a convenient environment to accomplish their applications. These systems, however, do not provide mathematical exactness for the verification of results nor include the newest theoretical results from the area. Consequently, knowledge in electrical engineering is often propagated by reusing experimental results that proved to be successfully. One reason is, that the use of exact mathematical results for these applications is too expensive to be explicitely performed. Furthermore — maybe also as a consequence of the above reason — there are theoretical results that could be advantageously used in applications but are not sufficiently known to electrical engineers.

In this situation mathematical knowledge management can contribute in two ways. Firstly, the widespread use of mathematical knowledge management systems incorporating electrical engineering could lead to a rediscovering and broader use of theoretical results in applications by electrical engineers. Secondly, the support in using these results could help filling the gap between fundamentals and applications in the sense that more new applications are based on mathematical fundamentals.

In this paper we focus on network theory [Unb93], in particular on network stability. Network theory deals with the mathematical description, analysis, and synthesis of electrical (continous and time-discrete) networks. For a realible application such systems have to be stable, that is for an arbitrary (bounded) input the output have to be bounded again. In case of highly-precise filters, however, it turns out that checking for stability is often hard to accomplish numerically. In this situation for example Schur's theorem [Sch21] permits an easy method to decide whether a network is stable by computing a chain of polynomials with decreasing degrees. We shall discuss the mathematical fundamentals and prequisites of Schur's theorem and present a Mizar formalization of this theorem.

The plan of the paper is as follows. In the next section we give a brief introduction to network theory focusing on the stability of networks and Schur's theorem [Sch21]. Then after a short review of the Mizar system [Miz07] we present our formalization of Schur's theorem in section 4. Finally, we discuss our results, draw conclusions for mathematical knowledge management in electrical engineering and give some hints for further work.

2 Networks and their Stability

As mentiond in the introduction the stability of networks is one of the main issues when dealing with the analysis and design of electrical circuits and systems. In the following we briefly review definitions and properties of electrical systems necessary to understand the application of Schur's theorem to electrical networks. In electrical engineering stability applies to the input/output behaviour of networks (see figure 1). For (time-) continous systems one finds the following definition. For discrete systems an analogous definition is used.

Definition 1. ([Unb93])
A continous system is (BIBO-)[3] stable, if and only if each bounded input signal $x(t)$ results in a bounded output signal $y(t)$.

[3] BIBO stands for Bounded Input Bounded Output.

Physically realizable, linear time-invariant systems (LTI systems) can be described by a set of linear equations [Unb93]. The behaviour of a LTI system then is completely characterized by its impulse response $h(t)$.[4] If the impulse response of auch a system is known, the relation between the input $x(t)$ and the output $y(t)$ is given by the convolution integral

$$y(t) \;=\; \int\limits_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau. \tag{1}$$

Furthermore, a LTI system is stable, if and only if its impulse response $h(t)$ is absolute integrable, that is there exists a constant K such that

$$\int\limits_{-\infty}^{\infty} |h(\tau)|\,\mathrm{d}\tau \;\leq\; K \;<\; \infty. \tag{2}$$

In network and filter analysis and design, however, one commonly employs the frequency domain rather than the time domain. To this end the system is described based on its transfer function $H(s)$. In case the Laplace transformation is used we have[5]

$$H(s) \;=\; \int\limits_{-\infty}^{\infty} h(t)e^{-st}\mathrm{d}t. \tag{3}$$

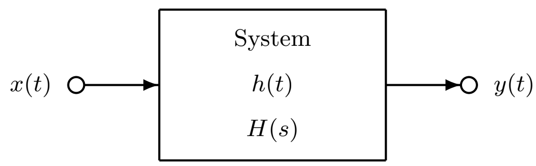

Figure 1: LTI system with one input $x(t)$ and one output $y(t)$

The evaluation of $H(s)$ for $s = j\omega$ — in case of convergence — enables the qualitative understanding of how the system handles and selects various frequencies, so for example whether the system describes a high-pass filter, low-pass filter, etc. Now the necessary condition to demonstrate the stability of LTI systems in the frequency domain reduces to show, that the $j\omega$-axis lies in the Laplace transformation's region of convergence (ROC).

[4] $h(t)$ is the output of the system, when the input is the Dirac delta function $\delta(t)$.
[5] Note that this is a generalization of the continous-time Fourier transformation.

For physically realizable LTI systems, such as the class of networks with constant and concentrated parameters, $H(s)$ is given in form of a rational function with real coefficients, that is

$$H(s) \;=\; \frac{a_n s^n + \ldots + a_0}{b_m s^m + \ldots + b_0}, \quad a_i, b_i \in \mathbb{R}. \tag{4}$$

In this case the region of convergence can be described by the roots of the denominator polynomial: If $s_i = \sigma_i + j\omega_i$ for $i = 1, \ldots m$ are the roots of $b_m s^m + \ldots + b_0$, the region of convergence is given by

$$\Re\{s\} \; > \max\{\sigma_i, \; i = 1, \ldots m\}.$$

To check stability it is therefore sufficient, to show that the real part $\Re\{s\}$ of all poles of $H(s)$ is smaller then 0. The denominator of $H(s)$ is thus a so-called Hurwitz polynomial.

The stability problem for discrete-time signals and systems can be analized with the same approach. For a given discrete-time transfer function $H(z)$ in the Z- domain, it has to be checked whether the unit circle is contained in the region of convergence. Hence for all poles z_i of $H(z)$ we must have $|z_i| < 1$. Using bilinear transformations [OS98]

$$z \;:=\; \frac{1+s}{1-s}. \tag{5}$$

it is thus sufficient to check whether the denominator of

$$H(z)|_{z:=\frac{1+s}{1-s}} \tag{6}$$

is a Hurwitz polynomial.

The practical examination of stability of highly-precise filters, however, turns out to be very hard. In practical applications the poles of concern are usually close to the axis $s = j\omega$ or the unit circle $|z| = e^{j\omega}$ respectively. Thus numerical determination of the poles is highly error-proning due to its rounding effects. In digital signal processing in addition degrees of transfer functions tend to be very high, for example 128 and higher in communication networks.

It is here that the theorem of Schur [Sch21] comes into play. Using the conjugate polynomial

$$f^*(x) \;:=\; a_0^* - a_1^* \, x + a_2^* \, x^2 - \ldots + (-1)^n a_n^* \, x^n \tag{7}$$

of a complex polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n$ a polynomial $g(x)$ of smaller degree is constructed, so that $g(x)$ is a Hurwitz polynomial if and only if $f(x)$ is. The construction itself is fairly easy: it is essentially a division by a linear polynomial.

Theorem 1. ([Sch21])
Let $\Re\{\xi\} < 0$. Then $f(x)$ is a Hurwitz polynomial if and only if $|f(\xi)| < |f^*(\xi)|$ and

$$g(x) \;:=\; \frac{f^*(\xi)f(x) - f(\xi)f^*(x)}{x - \xi}$$

is a Hurwitz polynomial.

The fact that the degree of $g(x)$ is strictly smaller than the one of $f(x)$ then allows to check stability of networks without explicitely computing roots of polynomials. Note that in addition ξ can always be chosen as -1, so that division can actually be performed by shifting. This however is not widely known in the area of network theory and we are not aware of any system using Schur's theorem for performing stability checks.

3 The Mizar System

The logical basis of Mizar [RT01,Miz07] is classical first order logic extended, however, with so-called schemes. Schemes introduce free second order variables, in this way enabling amongothers the definition of induction schemes. In addition Mizar objects are typed, the types forming a hierarchy with the fundamental type `set`. The user can introduce new (sub)types describing mathematical objects such as groups, fields, vector spaces or polynomials over rings or fields. To this end the Mizar language provides a powerful typing mechanism based on adjective subtypes [Ban03].

The current development of the Mizar Mathematical Library (MML) relies on Tarski-Grothendieck set theory — a variant of Zermelo Fraenkel set theory using Tarski's axiom on arbitrarily large, strongly inaccessible cardinals [Tar39] which can be used to prove the axiom of choice —, though in principle the Mizar language can be used with other axiom systems also. Mizar proofs are written in natural deduction style as presented in the calculus of [Jaś34]. The rules of the calculus are connected with corresponding (English) natural language phrases so that the Mizar language is close to the one used in mathematical textbooks. The Mizar proof checker verifies the individual proof steps using the notion of obvious inferences [Dav81] to shorten the rather long proofs of pure natural deduction.

The basic theories necessary for Schur's theorem are already contained in MML: Polynomials (over arbitrary rings) have been defined in [Mil01b]. The original goal here was to prove the fundamental theorem of algebra. The complex numbers have been introduced in [Byl90] as objects in their on right. To use the theory of polynomials we need, however, the ring structure of complex numbers. Fortunately, this has been established in [Mil01a]. Consequently, using Mizar we were able to apply — besides the theory of polynomials — both general ring (or field) theorems for complex numbers and special theorems valid for complex numbers only.

4 Mizar Formalization of Schur's theorem

4.1 Some Preliminaries About Polynomials

Although the theory of polynomials in Mizar is rather well developed, division of polynomials had not been introduced, yet. This, however, can be done (for arbitray fields) in a straightforward way following the well-known literature.[6] We defined two functors `div` and `mod` for the quotient and the remainder, respectively. The keyword `it` denotes the object being defined. Note that Mizar requires an existence and a uniqueness proof for functors. Here, however, these have to be performed for the first definition only, because the definition of `mod` employs solely arithmetics of polynomials — including the just defined functor `div`. Therefore existence and uniqueness in this case is automatically derived by the Mizar checker.

```
definition
let L be Field;
let p,q be Polynomial of L such that q <> 0_.(L);
func p div q -> Polynomial of L means
  ex r being Polynomial of L st p = it *' q + r & deg r < deg q;
end;

definition
let L be Field;
let p,q be Polynomial of L such that s <> 0_.(L);
func p mod q -> Polynomial of L equals
  p - (p div q) *' q;
end;
```

Divisibility of polynomials can then be introduced by the condition `p mod q = 0._(L)`, where `0._(L)` is the zero polynomial, or by the equivalent condition that there exists a polynomial `h` such that `p = h * q`. For our purposes it is essential that a polynomial $p(x)$ is divisible without remainder by the linear polynomial $x - z$, if z is a root of $p(x)$.[7] It was therefore necessary to show that for every root z of a polynomial $p(x)$ the polynomial $x-z$ is a divisor of $p(x)$. To do so, we introduced the polynomials `rpoly(k,z)` $= x^k - z^k$ and `qpoly(k,z)` $= x^{k-1} + x^{k-2} * z + x^{k-3} * z^2 + ... + x * z^{k-2} + z^{k-1}$. Note that for `k` > 1 we have `rpoly(1,z) * qpoly(k,z) = rpoly(k,z)`, which allows for the construction of a polynomial `h` such that `r(1,z) * h = p`. We thus get

```
theorem
for L being Field
for p being Polynomial of L
for z being Element of L st z is_a_root_of p holds rpoly(1,z) divides p;
```

Note again, that this property is shown for polynomials over arbitrary fields. In the next section when dealing with Schur's criterium, we shall use the complex number version of this theorem.

[6] see for example [GG99].

[7] compare theorem 1 in section 2.

4.2 Schur's Theorem

Using the general Mizar theory of polynomials for our purposes, that is for polynomials over the complex numbers, is straightforward. We just instantiate the parameter `L` describing the coefficient domain with the field of complex numbers `F_Complex` from [Mil01a]. So an object of type

```
Polynomial of F_Complex
```

combines the theory of polynomials with the one of complex numbers. Hence for such objects we have available both the predicate `is_root_of` defined for polynomials and the functor `Re` giving the real part of a complex number. This allows for the following definition of Hurwitz polynomials.

```
definition
let f be Polynomial of F_Complex;
attr f is Hurwitz means
  for z being Element of F_Complex st z is_a_root_of f holds Re(z) < 0;
end;
```

The examination of polynomials with a degree smaller or equal then 1 is rather uncomplex. Constant polynomials are not Hurwitz, except for the zero polynomial which is. A linear polynomial $p(x) = x - z$ obviously is Hurwitz if and only if the real part of z is smaller than 0. This condition carries over to arbitrary polynomials of degree 1. Hence we get the following three theorems for the basic cases.

```
theorem
0_.(F_Complex) is non Hurwitz;

theorem
for z being Element of F_Complex st z <> 0.F_Complex
holds z * 1_.(F_Complex) is Hurwitz;

theorem
for z1,z2 being Element of F_Complex st z1 <> 0.F_Complex
holds z1 * rpoly(1,z2) is Hurwitz iff Re(z2) < 0;
```

In addition we proved some other properties of Hurwitz polynomials needed later, so for example that $f * g$ is Hurwitz if and only if f and g are Hurwitz or that for a complex number $z \neq 0$ we have $z * f$ is Hurwitz if and only if f is Hurwitz.

To prove Schur's theorem for the general case we needed to introduce the conjugate of a complex polynomial as given by equation (7). This is accomplished by a Mizar functor `*'` defining the coefficients of the conjugated polynomial appropriately.[8] For that we use the functor `power(G)` which describes exponentiation

[8] Note that the functor `*'` is then overloaded, because it also stands for conjugation of complex numbers as can be seen in the following definition.

for arbitrary groups G, here again instantiated with F_Complex, the field of complex numbers. Note that after instantiating G with F_Complex the resulting type of the functor power(F_Complex) is automatically accomodated, so that it is no problem multiplying its result with another complex number.

```
definition
let f be Polynomial of F_Complex;
func f*' -> Polynomial of F_Complex means
  for i being Element of NAT holds
  it.i = power(F_Complex).(-1.F_Complex,i) * (f.i)*';
end;
```

Thus prepared we could already state Schur's theorem in Mizar. However, to shorten writings we decided to introduce another functor describing the nominator polynomial of Schur's construction. The functor eval describes evaluation of polynomials.

```
definition
let f be Polynomial of F_Complex;
let z be Element of F_Complex;
func F*(f,z) -> Polynomial of F_Complex equals
  eval(f*',z) * f - eval(f,z) * f*';
end;
```

Taking into account that the Mizar functor |. .| gives the absolute value of complex numbers, we then get the following formulation of Schur's theorem. Note again that rpoly(1,z) is the polynomial $p(x) = x - z$.

```
theorem
for f being Polynomial of F_Complex st deg(f) >= 1
for z being Element of F_Complex
    st Re(z) < 0 & |.eval(f,z).| < |.eval(f*',z).|
holds f is Hurwitz iff F*(f,z) div rpoly(1,z) is Hurwitz;
```

The proof of the theorem relies on a thorough examination of the relation between the real part $\Re(z)$ of a complex number z and the values of $|f(z)|$ and $|f^*(z)|$ in case f is a Hurwitz polynomial. It turns out that whether $\Re(z)$ is smaller or greater than 0 completely determines which value $|f(z)|$ or $|f^*(z)|$ is greater. This allows later to argue about the roots of the nominator polynomial, that is of the polynomial F*(f,z).

```
theorem
for f being Polynomial of F_Complex st deg(f) >= 1 & f is Hurwitz
for z being Element of F_Complex
holds (Re(z) < 0 implies |.eval(f,z).| < |.eval(f*',z).|) &
      (Re(z) > 0 implies |.eval(f,z).| > |.eval(f*',z).|) &
      (Re(z) = 0 implies |.eval(f,z).| = |.eval(f*',z).|);
```

The corresponding proof is rather technical. In Mizar, however, the application of theorems for complex numbers has been automatized in the sense that a number of basic theorems are automatically applied, in this way shortening the proof [NB04]. In addition the Encyclopedia of Mathematics in Mizar (EMM) collecting theorems of a theory — in this case concerning complex numbers — originally spread over the whole repository produced a kindly working environment to accomplish the task.

Note also that this theorem implies that even for polynomials with degree > 1, it is not always necessary to reduce the problem of stability to a basic case: If we find a complex number z with $\Re(z) < 0$ such that $|f(z)| \geq |f^*(z)|$ we immediately get that f is not a Hurwitz polynomial.

```
theorem
for f being Polynomial of F_Complex st deg(f) >= 1
holds (ex z being Element of F_Complex
       st Re(z) < 0 & |.eval(f,z).| >= |.eval(f*',z).|)
implies f is non Hurwitz;
```

The rest of the proof basically applies the theorem from above two times, once for each direction. We first proved the following, more general version of Schur's theorem from [Sch21]: For complex numbers z_1 and z_2 such that $|z_1| > |z_2|$ and a complex polynomial $f(x)$ with degree ≥ 1 holds $f(x)$ is a Hurwitz polynomial if and only if $g(x) = z_1 * f(x) - z_2 * f^*(x)$ is a Hurwitz polynomial: Because of $|z_1| > |z_2|$ we have now $|f(x)| \geq |f^*(x)|$, if $\Re(x) \geq 0$, and hence $|z_1 * f(x)| > |z_2 * f^*(x)|$, which shows the first direction. For the other direction we only note, that $f(x) = z_1' * g(x) - z_2' * g^*(x)$ with

$$z_1' \;=\; \frac{z_1^*}{|z_1|^2 - |z_2|^2} \quad \text{and} \quad z_2' \;=\; -\frac{z_2}{|z_1|^2 - |z_2|^2},$$

so that $|z_1'| > |z_2'|$ finishes the proof.

From this Schur's theorem easily follows by instantiating z_1 with $f^*(z)$ and z_2 with $f(z)$ giving essentially the functor `F*(f,z)` from above. Note that we here need in addition that the denominator polynomial $p(x) = x - z$, that is `rpoly(1,z)`, divides the nominator polynomial $f^*(z) * f(x) - f(z) * f^*(x)$, that is `F*(f,z)`. This, however, is ensured by the fact that z is a root of $f^*(z) * f(x) - f(z) * f^*(x)$ and the — automatically available — complex number version of the main theorem of section 4.1.

So this part of the proof requires both arithmetics — including conjugates — and abstract values of complex numbers and arithmetics of polynomials over complex numbers. In Mizar, as already mentioned, this is achieved by instantiating the general theory of polynomials with the field of complex numbers. Then of course the absolute value, defined originally for complex numbers, is available for the coefficients of complex polynomials, also. Consequenly, the just described proof steps could be accomplished based on these two theories without other preparations or additional lemmas.

5 Conclusions

In this paper we have considered electrical engineering as an application area for mathematical knowledge management. We have focused on stability theory of networks and have shown by a Mizar formalization of Schur's theorem that interesting mathematical knowledge in electrical engineering can be successfully handled with mathematical knowledge management systems.

We believe that both electrical engineering and mathematical knowledge management can benefit from a further development of collaboration in the area of mathematical knowledge. The combination of mathematical knowledge managements systems and repositories with analysis and design tools for electrical networks can provide electrical engineers with a thorough mathematical basis for their work. In addition this would lead also to the use of less known theoretical results, such as for example Schur's theorem, in new applications.

For mathematical knowledge management electrical engineering can serve as an additional test bed, in which new developments can be tried out. And, of course, in this way a whole group of potential new users of mathematical knowledge management systems could be adressed.

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