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A NUMERICAL METHOD OF BICHARACTERISTICS FOR QUASI-LINEAR PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract — Classical solutions of mixed problems for first order partial functional differential equations in several independent variables are approximated by solutions of an Euler-type difference problem. The mesh for the approximate solutions is obtained by the numerical solution of equations of bicharacteristics. The convergence of explicit difference schemes is proved by means of consistency and stability arguments. It is assumed that the given functions satisfy the nonlinear estimates of the Perron type. Differential systems with deviated variables and differential integral systems can be obtained from the general model by specializing the given operators.

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1. Introduction

For any metric spaces X and Y, we denote by $C(X,Y)$ the class of all continuous functions from X into Y. We will use inequalities between vectors, understanding that the same inequalities hold between their corresponding components.

Let $a>0$, $d_0 \in \mathbb{R}_+$, $\mathbb{R}_+ = [0,+\infty)$, $d=(d_1,\dots,d_n) \in \mathbb{R}^n_+$ and $b=(b_1,\dots,b_n) \in \mathbb{R}^n$ be given where $b_j>0$ for $1 \leqslant j \leqslant n$. We define the sets

$$E=[0,a]\times[-b,b], \quad D=[-d_0,0]\times[-d,d].$$

Let $c=(c_1,\dots,c_n)=b+d$ and

$$E_0=[-d_0,0]\times[-c,c], \quad \partial_0 E=[0,a]\times([-c,c]\setminus(-b,b)),$$

$$\Omega=E_0\cup E\cup\partial_0 E, \quad \Xi=E\times C(D,\mathbb{R}).$$

Suppose that $z:\Omega\to\mathbb{R}$ and $(t,x)\in E$ are fixed. We define the function $z_{(t,x)}:D\to\mathbb{R}$ as follows:

$$z_{(t,x)}(\xi,y)=z(t+\xi,x+y), \quad (\xi,y)\in D.$$

The function $z_{(t,x)}$ is the restriction of z to the set $[t-d_0,t]\times[x-d,x+d]$ and this restriction is shifted to the set D. Elements of the space $C(D,\mathbb{R})$ will be denoted by $w,\bar{w}$ and so on. We will write $\|\cdot\|_D$ for the maximum norm in the space $C(D,\mathbb{R})$. Let

$$f:\Xi\to\mathbb{R}^n, \quad f=(f_1,\dots,f_n), \quad G:\Xi\to\mathbb{R}, \quad \varphi:E_0\cup\partial_0E\to\mathbb{R},$$

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$$\alpha_0 : E \to \mathbb{R}, \quad \alpha' : E \to \mathbb{R}^n, \quad \alpha' = (\alpha_1, \ldots, \alpha_n)$$

be given functions. Write $\alpha(t,x) = (\alpha_0(t,x), \alpha'(t,x))$, $(t,x) \in E$. We require that $\alpha(t,x) \in E$ and $\alpha_0(t,x) \leqslant t$ for $(t,x) \in E$. We consider a problem consisting of the functional differential equation

$$\partial_t z(t,x) + \sum_{j=1}^{n} f_j(t,x,z_{\alpha(t,x)}) \partial_{x_j} z(t,x) = G(t,x,z_{\alpha(t,x)}) \tag{1.1}$$

and the initial-boundary condition

$$z(t,x) = \varphi(t,x) \quad \text{on} \quad E_0 \cup \partial_0 E. \tag{1.2}$$

A function $v : \Omega \to \mathbb{R}$ is called a classical solution of the above problem if:
i) $v \in C(\Omega, \mathbb{R})$ and v is of class C^1 on E,
ii) v satisfies equation (1.1) on E and initial boundary condition (1.2) holds.
We are interested in solving problem (1.1), (1.2)numerically.
Write

$$\Delta_+^{(j)} = \{(t,x) \in E : \; x_j = b_j\}, \qquad \Delta_-^{(j)} = \{(t,x) \in E : \; x_j = -b_j\},$$

where $1 \leqslant j \leqslant n$, and

$$\Delta = \bigcup_{j=1}^{n} \left(\Delta_+^{(j)} \cup \Delta_-^{(j)} \right).$$

We will assume that $f \in C(\Xi, \mathbb{R}^n)$, $G \in C(\Xi, \mathbb{R})$, $\varphi \in C(E_0 \cup \partial_0 E, \mathbb{R})$ and that $f_j < 0$ on $\Delta_+^{(j)} \times C(D, \mathbb{R})$ and $f_j > 0$ on $\Delta_-^{(j)} \times C(D, \mathbb{R})$ for $1 \leqslant j \leqslant n$. Note that if this condition is satisfied, then under natural assumptions on regularity of f, G and φ there exists a classical solution of (1.1), (1.2) and it is unique.

Note that our hereditary setting contains the well-known delay structures as particular cases.

Example 1.1. *Suppose that $\tilde{f}$, $\tilde{G} : \Xi \to \mathbb{R}$ are given functions. If we set $f(t,x,w) = \tilde{f}(t,x,w(0,0))$ and $G(t,x,w) = \tilde{G}(t,x,w(0,0))$, then*

$$f(t,x,z_{\alpha(t,x)}) = \tilde{f}(t,x,z(\alpha(t,x))), \quad G(t,x,z_{\alpha(t,x)}) = \tilde{G}(t,x,z(\alpha(t,x)))$$

and (1.1) *becomes an equation with deviated variables.*

Example 1.2. *Assume that β, $\gamma : E \to \mathbb{R}^{1+n}$. For the above $\tilde{f}$, $\tilde{G}$ we put*

$$f(t,x,w) = \tilde{f}\Bigg(t,x, \int\limits_{(\beta-\alpha)(t,x)}^{(\gamma-\alpha)(t,x)} w(\xi,y)\, dy\, d\xi\Bigg), \quad G(t,x,w) = \tilde{G}\Bigg(t,x, \int\limits_{(\beta-\alpha)(t,x)}^{(\gamma-\alpha)(t,x)} w(\xi,y)\, dy\, d\xi\Bigg),$$

then

$$f(t,x,z_{\alpha(t,x)}) = \tilde{f}\Bigg(t,x, \int\limits_{\beta(t,x)}^{\gamma(t,x)} w(\xi,y)\, dy\, d\xi\Bigg), \quad G(t,x,z_{\alpha(t,x)}) = \tilde{G}\Bigg(t,x, \int\limits_{\beta(t,x)}^{\gamma(t,x)} w(\xi,y)\, dy\, d\xi\Bigg)$$

and (1.1) *becomes a differential integral equation.*

In recent years, a number of papers concerning numerical methods for functional partial differential equations have been published. The difference methods and monotone iterative methods for nonlinear parabolic problems were studied in [9, 10]. The quasi-linear first order functional differential systems and the general class of difference schemes with suitable interpolation operators were considered in [4]. The monograph [6] contains an exposition of the theory of difference methods for hyperbolic functional differential problems. The main problem in these investigations is to find a difference functional equation which is stable and satisfies the consistency conditions with respect to the original problem with sufficiently regular functions. A comparison technique is used in the investigation of the stability of functional difference problems.

We use in the present paper general ideas concerning numerical methods for first order partial differential equations, which were introduced in [3, 6, 8]. The numerical method of bicharacteristics for quasi-linear hyperbolic systems was treated in [7]. The unknown function of only two independent variables was considered in that paper; this constraint can be omitted, by the proper choice of approximation, which was first proposed in [1].

Results for quasi-linear hyperbolic functional differential problems can be found in [2, 6] (Chapter 3).

First order partial differential equations with deviated variables and differential integral equations find applications in different fields of knowledge. Examples of such applications can be found in [1, 6].

Our motivation to investigate the numerical methods of bicharacteristics is as follows.

Two types of assumptions are needed in the theorem on the stability of classical difference schemes corresponding to (1.1), (1.2). The first type conditions concern regularity of f and G. It is assumed that f and G satisfy Perron type estimates with respect to the functional variable. The second type conditions concern the mesh. It is required that

$$\frac{1}{n} - \frac{h_0}{h_j}|f_j(P)| \geqslant 0 \quad \text{for} \quad 1 \leqslant j \leqslant n, \quad P \in \Xi. \tag{1.3}$$

where h_0 and $(h_1, \ldots, h_n)$ are steps of the mesh with respect to t and $(x_1, \ldots, x_n)$. The above assumption is known as the generalized Courant — Friedrichs — Lewy condition for (1.1), (1.2).

We show that there are numerical methods for (1.1), (1.2) which are convergent and assumption (1.3) is omitted.

2. Discretization of mixed problems

Let us denote by $\mathbf{F}(X, Y)$ the class of all functions defined on X and taking values in Y, where X and Y are arbitrary sets. Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of natural numbers and integers, respectively. Let us fix our notations on vectors. For $x, y \in \mathbb{R}^n$, where

$$x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_n),$$

we put

$$\|x\| = \sum_{j=1}^{n} |x_j|, \quad x \diamond y = (x_1 y_1, \ldots, x_n y_n).$$

Consider the set Ω. Let (h_0, h'), $h' = (h_1, \ldots, h_n)$, stand for steps of the mesh. For $h = (h_0, h')$ and $(r, m) \in \mathbb{Z}^{1+n}$ where $m = (m_1, \ldots, m_n)$, we define

$$t^{(r)} = rh_0, \quad x^{(m)} = m \diamond h', \quad x^{(m)} = (x_1^{(m_1)}, \ldots, x_n^{(m_n)}).$$

Let us denote by H the set of all $h = (h_0, h')$ such that there are $K_0 \in \mathbb{N}$, $N \in \mathbb{N}^n$ with the properties $K_0 h_0 = d_0$ and $N \diamond h' = b$. Let $K \in \mathbb{N}$ be defined by the relation $Kh_0 \leqslant a < (K+1)h_0$. For $h \in H$ we put $\|h\| = h_0 + h_1 + \ldots + h_n$. Write

$$\mathbb{R}_h^{1+n} = \{(t^{(r)}, x^{(m)}) : (r, m) \in \mathbb{Z}^{1+n}\}, \quad I_h = \{t^{(r)} : 0 \leqslant r \leqslant K\},$$

$$E_{0.h} = E_0 \cap \mathbb{R}_h^{1+n}, \quad \partial_0 E_h = \partial_0 E \cap \mathbb{R}_h^{1+n}.$$

For X_h being an arbitrary subset of $\mathbb{R}_h^{1+n}$ and for the functions $z_h : X_h \to \mathbb{R}$ and $\eta : I_h \to \mathbb{R}$ we write $z_h^{(r,m)} = z(t^{(r)}, x^{(m)})$ and $\eta^{(r)} = \eta(t^{(r)})$. We put

$$\Omega^{(r)} = \Omega \cap \left([-d_0, t^{(r)}] \times \mathbb{R}^n\right)$$

and

$$\|z\|_{\Omega^{(r)}} = \max\left\{ |z(t,x)| \ : \ (t,x) \in \Omega^{(r)} \right\} \quad \text{for} \quad z \in \mathbf{F}(\Omega, \mathbb{R}) \quad \text{and} \quad 0 \leqslant r \leqslant K.$$

Set

$$S_h = (E_{0.h} \cup \partial_0 E_h) \cap E.$$

Now we define a few sets of indexes:

$$\Sigma = \{\sigma' = (\sigma_1, \ldots, \sigma_n), \ \ \sigma_k \in \{0, 1\} \ \text{ for } \ 1 \leqslant k \leqslant n\}, \tag{2.1}$$

$$\tilde{\Sigma} = \{(\sigma_0, \sigma') : \sigma_0 \in \{-1, 0\}, \ \ \sigma' \in \Sigma\},$$

and

$$\Sigma^{(s,m)} = \{(\sigma_0, \sigma') \in \tilde{\Sigma} : (t^{(s+\sigma_0)}, x^{(m+\sigma')}) \in S_h\}, \quad (t^{(s)}, x^{(m)}) \in S_h. \tag{2.2}$$

Denote

$$\Theta = \{\theta \in \mathbf{F}(H, \mathbb{R}_+) : \lim_{h \to 0} \theta(h) = 0\}.$$

The numerical method of bicharacteristics consists in replacing problem (1.1), (1.2) by the system of difference equations for unknown functions

$$\eta = (\eta_1, \ldots, \eta_n) \quad \text{and} \quad z.$$

Now we present these difference equations.

The functional differential problems considered in the present paper have the following property. Equation (1.1) contains the functional variable $z_{\alpha(t,x)}$ which is an element of the space $C(D, \mathbb{R})$. Numerical solutions of (1.1), (1.2) are functions defined on finite sets. Therefore we need an approximation operator $\widehat{T}_h$ in the system of difference equations.

Suppose that $(t^{(s)}, x^{(m)}) \in S_h$, $0 \leqslant s \leqslant K-1$, and $\varphi_h : E_{0.h} \cup \partial_0 E_h \to \mathbb{R}$ is a given function. We define

$$\eta^{(s)} = x^{(m)}, \quad z^{(s,m)} = \varphi_h^{(s,m)} \tag{2.3}$$

and

$$\eta^{(r+1)} = \eta^{(r)} + h_0 f(t^{(r)}, \eta^{(r)}, (\widehat{T}_h z)_{\alpha(t^{(r)}, \eta^{(r)})}), \tag{2.4}$$

$$z(t^{(r+1)},\eta^{(r+1)}) = z(t^{(r)},\eta^{(r)}) + h_0 G(t^{(r)},\eta^{(r)},(\widehat{T}_h z)_{\alpha(t^{(r)},\eta^{(r)})}), \tag{2.5}$$

for $s \leqslant r \leqslant K-1$. Let us denote by $(g_h(\cdot,t^{(s)},x^{(m)}), z_h)$ a solution of problem (2.3)–(2.5), where $g_h = (g_{h.1},\dots,g_{h.n})$. We write

$$\widehat{E}_h = \{(t^{(r)}, g_h(t^{(r)},t^{(s)},x^{(m)})) \in E : (t^{(s)},x^{(m)}) \in S_h, \quad s \leqslant r \leqslant K\},$$

$$\widehat{\Omega}_h = \widehat{E}_h \cup E_{0.h} \cup \partial_0 E_h, \quad \widehat{\Omega}_h^{(r)} = \widehat{\Omega}_h \cap ([-d_0,t^{(r)}] \times \mathbb{R}^n), \quad 0 \leqslant r \leqslant K.$$

Remark 2.1. Note that in the numerical method of characteristics (see [1]) we are able to construct the whole mesh before we start to compute the approximate solution. The situation is different in the method of bicharacteristics (2.3)–(2.5), because the location of nodes $(t^{(r+1)},x) \in \widehat{\Omega}_h$ depends on the values of the approximate solution z_h on the set $\widehat{\Omega}_h^{(r)}$. Moreover, the approximation operator $\widehat{T}_h$, which we define further, depends on the mesh $\widehat{\Omega}_h$. Hence the approximate solution z_h, the mesh $\widehat{\Omega}_h$, and the approximation operator $\widehat{T}_h$ should be controlled simultaneously in the course of our computations.

2.1. Notation of error. Suppose that $v : \Omega \to \mathbb{R}$ is of class C^1. We will denote by

$$g[v](\cdot,t,x) = (g_1[v](\cdot,t,x),\dots,g_n[v](\cdot,t,x))$$

the set of bicharacteristics of equation (1.1) corresponding to v. Then the function $g[v](\cdot,t,x)$ is the solution of the Cauchy problem

$$w'(\xi) = f(\xi, w(\xi), v_{\alpha(\xi,w(\xi))}), \quad w(t) = x, \tag{2.6}$$

where $f = (f_1,\dots,f_n)$. The numerical procedure (2.3)–(2.5) generates two sets of functions: $\{g_h\}_{h\in H}$ and $\{z_h\}_{h\in H}$. The function g_h is used to constructe the set $\widehat{\Omega}_h$ whereas z_h is considered as an approximate solution to problem (1.1), (1.2) and $z_h \in \mathbf{F}(\widehat{\Omega}_h,\mathbb{R})$. The idea of the proof of convergence, together with the content of Remark 2.1, leads to the following notation of the error of the method (2.3)–(2.5):

$$[|v - z_h|]_{h.E}^{(r)} = \max\{|v(t^{(i)}, g[v](t^{(i)},t,x)) - z_h(t^{(i)}, g_h(t^{(i)},t,x))| : (t,x) \in S_h,\ t \leqslant t^{(i)} \leqslant t^{(r)}\},$$

$$[|v - z_h|]_{h.\partial}^{(r)} = \max\{|v(t^{(i)},x^{(m)}) - z_h(t^{(i)},x^{(m)})| : (t^{(i)},x^{(m)}) \in E_{0.h} \cup \partial_0 E_h,\ i \leqslant r\},$$

$$[|v - z_h|]_h^{(r)} = \max\{[|v - z_h|]_{h.E}^{(r)},\ [|v - z_h|]_{h.\partial}^{(r)}\},$$

and

$$[|g[v] - g_h|]_h^{(r)} = \max\{\|g[v](t^{(i)},t,x) - g_h(t^{(i)},t,x)\| : (t,x) \in S_h,\ t \leqslant t^{(i)} \leqslant t^{(r)}\},$$

$$\delta\widehat{\Omega}_h^{(r)} = \frac{1}{2}\max\Big\{\|g_h(t^{(i)},t^{(s)},x^{(m)}) - g_h(t^{(i)},t^{(s+\sigma_0)},x^{(m+\sigma')})\| : \\ (t^{(s)},x^{(m)}) \in S_h,\ (\sigma_0,\sigma') \in \Sigma^{(s,m)},\ s \leqslant i \leqslant r\Big\},$$

where $\Sigma^{(s,m)}$ is defined by (2.2) and $0 \leqslant r \leqslant K$. Then the error of the method (2.3)–(2.5) is defined by

$$\varepsilon_h = \max\Big\{[|v - z_h|]_h,\ [|g[v] - g_h|]_h,\ 2\,\delta\widehat{\Omega}_h\Big\}. \tag{2.7}$$

We prove that for v satisfying (1.1), (1.2), for sufficiently regular f and G and for the approximation operator $\widehat{T}_h$ satisfying certain additional conditions, we have

$$\lim_{h\to 0} \varepsilon_h^{(r)} = 0 \quad \text{uniformly with respect to } r, \quad 0 \leqslant r \leqslant K.$$

2.2. Assumptions on the approximation operator.

Assumption $H[\widehat{T}_h]$. The operator $\widehat{T}_h : \mathbf{F}(\widehat{\Omega}_h, \mathbb{R}) \to \mathbf{F}(\Omega, \mathbb{R})$ satisfies the conditions:
1) if $w \in \mathbf{F}(\widehat{\Omega}_h, \mathbb{R})$ then $\widehat{T}_h w \in C(\Omega, \mathbb{R})$;
2) if $t^{(r)} \in I_h$ and w, $\bar{w} \in \mathbf{F}(\widehat{\Omega}_h, \mathbb{R})$ are such functions that

$$w|_{\widehat{\Omega}_h^{(r)}} = \bar{w}|_{\widehat{\Omega}_h^{(r)}}$$

then

$$\widehat{T}_h w|_{\Omega^{(r)}} = \widehat{T}_h \bar{w}|_{\Omega^{(r)}}$$

(Volterra condition);
3) for each $v \in C^1(\Omega, \mathbb{R})$ and for (g_h, z_h) satisfying (2.3)–(2.5) there is $\beta \in \Theta$ and C_1, $C_2 \in \mathbb{R}_+$ such that for $0 \leqslant r \leqslant K$

$$\|v - \widehat{T}_h z_h\|_{\Omega^{(r)}} \leqslant [|v - z_h|]_h^{(r)} + C_1[|g[v] - g_h|]_h^{(r)} + C_2\, \delta\widehat{\Omega}_h^{(r)} + \beta(h). \tag{2.8}$$

Examples of $\widehat{T}_h$, satisfying Assumption $H[\widehat{T}_h]$, are given in Section 4.

3. Convergence of the numerical method

The main assumptions on f, G and α are the following:

Assumption $H[f, G, \varphi]$. The functions $f : \Xi \to \mathbb{R}^n$ and $G : \Xi \to \mathbb{R}$ are continuous and there is $\sigma : [0, a] \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

1) σ is continuous and nondecreasing with respect to both variables,

2) $\sigma(t, 0) = 0$ for $t \in [0, a]$ and for each $\rho \geqslant 1$ the maximal solution of the Cauchy problem

$$\zeta'(t) = \sigma(t, \rho\zeta(t)), \quad \zeta(0) = 0 \tag{3.1}$$

is $\zeta(t) = 0$ for $t \in [0, a]$;

3) the estimates

$$\|f(t, x, w) - f(t, \bar{x}, \bar{w})\| \leqslant \sigma(t, \|x - \bar{x}\| + \|w - \bar{w}\|_D),$$

$$|G(t, x, w) - G(t, \bar{x}, \bar{w})| \leqslant \sigma(t, \|x - \bar{x}\| + \|w - \bar{w}\|_D)$$

are satisfied on Ξ,

4) there is $\delta = (\delta_1, \ldots, \delta_n)$, $\delta_j > 0$, $1 \leqslant j \leqslant n$, such that for each $(t, w) \in [0, a] \times C(D, \mathbb{R})$ and for each j, $1 \leqslant j \leqslant n$, we have

$$f_j(t, x, w) < 0 \quad \text{for} \quad x \in [-b, b], \quad x_j \geqslant b_j - \delta_j$$

and

$$f_j(t, x, w) > 0 \quad \text{for} \quad x \in [-b, b], \quad x_j \leqslant -b_j + \delta_j;$$

5) the functions $\alpha_0 \in C(E, \mathbb{R})$, $\alpha' \in C(E, \mathbb{R}^n)$ are such that $0 \leqslant \alpha_0(t, x) \leqslant t$, $\alpha'(t, x) \in [-b, b]$ for $(t, x) \in E$ and there is $p \in \mathbb{R}_+$ such that

$$|\alpha_0(t, x) - \alpha_0(t, \bar{x})| + \|\alpha'(t, x) - \alpha'(t, \bar{x})\| \leqslant p\|x - \bar{x}\| \quad \text{on} \quad E.$$

Theorem 3.1. *Suppose that Assumptions $H[\widehat{T}_h]$, $H[f,G,\varphi]$ are satisfied and*

1) *the function $\varphi : E_0 \cup \partial_0 E \to \mathbb{R}$ is of class C^1, the function $v : \Omega \to \mathbb{R}$ is the solution of (1.1), (1.2) and v is of class C^1 on Ω;*

2) *$h \in H$ and h_0 is so small that the inequality*

$$h_0 |f_j(t,x,w)| \leqslant \delta_j \quad \text{for} \quad (t,x,w) \in \Xi, \quad 1 \leqslant j \leqslant n,$$

holds true with $\delta = (\delta_1, \ldots, \delta_n)$ from Assumption $H[f,G,\varphi]$;

3) *(g_h, z_h) is the solution of (2.3)–(2.5) and there is $\psi_0 \in \Theta$ such that*

$$\|\varphi^{(r,m)} - \varphi_h^{(r,m)}\| \leqslant \psi_0(h) \quad \text{on} \quad E_{0.h} \cup \partial_0 E_h.$$

Then there is $\varepsilon_0 > 0$ and $\psi \in \Theta$ such that for $\|h\| < \varepsilon_0$ and $0 \leqslant r \leqslant K$ we have

$$\max\left\{ [|v - z_h|]_h^{(r)},\ [|g[v] - g_h|]_h^{(r)},\ 2\,\delta\widehat{\Omega}_h^{(r)} \right\} \leqslant \psi(h), \tag{3.2}$$

where $g[v] = (g_1[v], \ldots, g_n[v])$ is the set of bicharacteristics of equation (1.1).

Proof. The above condition 2) is sufficient for the numerical bicharacteristics to remain in the set E, i.e. for any $(t,x) \in S_h$ and $0 \leqslant r \leqslant K$ we have $g_h(t^{(r)}, t, x) \in [-b, b]$. This, together with Assumption $H[\widehat{T}_h]$, gives the existence and uniqueness of solutions for the problem (2.3)–(2.5).

Write $\varepsilon_{h.0} = [|v - z_h|]_h$, $\varepsilon_{h.1} = [|g[v] - g_h|]_h$ and $\varepsilon_{h.2} = 2\,\delta\widehat{\Omega}_h$. We will construct a difference inequality for the function $\varepsilon_h = \max\{\varepsilon_{h.0}, \varepsilon_{h.1}, \varepsilon_{h.2}\}$. Suppose that $(t,x) \in S_h$. Once (t,x), v are fixed, let us denote $g = g[v](\cdot, t, x)$ and $g_h = g_h(\cdot, t, x)$. Then we have

$$v(t^{(r+1)}, g(t^{(r+1)})) = v(t^{(r)}, g(t^{(r)})) + \int_{t^{(r)}}^{t^{(r+1)}} G(s, g(s), v_{\alpha(s,g(s))})ds, \tag{3.3}$$

and

$$g(t^{(r+1)}) = g(t^{(r)}) + \int_{t^{(r)}}^{t^{(r+1)}} f(s, g(s), v_{\alpha(s,g(s))})ds. \tag{3.4}$$

It follows from (2.3)–(2.5), that

$$z_h(t^{(r+1)}, g_h^{(r+1)}) = z_h(t^{(r)}, g_h^{(r)}) + h_0 G(t^{(r)}, g_h^{(r)}, (\widehat{T}_h z_h)_{\alpha(t^{(r)}, g_h^{(r)})}), \tag{3.5}$$

and

$$g_h^{(r+1)} = g_h^{(r)} + h_0 f(t^{(r)}, g_h^{(r)}, (\widehat{T}_h z_h)_{\alpha(t^{(r)}, g_h^{(r)})}). \tag{3.6}$$

First we write a difference inequality for $\varepsilon_{h.0}$. Subtracting (3.3) and (3.5) we find that

$$v(t^{(r+1)}, g(t^{(r+1)})) - z_h(t^{(r+1)}, g_h(t^{(r+1)})) = v(t^{(r)}, g(t^{(r)})) - z_h(t^{(r)}, g_h(t^{(r)})) +$$
$$h_0 \left[G(t^{(r)}, g(t^{(r)}), v_{\alpha(t^{(r)}, g(t^{(r)}))}) - G(t^{(r)}, g_h(t^{(r)}), (\widehat{T}_h z_h)_{\alpha(t^{(r)}, g_h(t^{(r)}))}) \right] + \Gamma_h^{(r)}, \tag{3.7}$$

where

$$\Gamma_h^{(r)} = \int_{t^{(r)}}^{t^{(r+1)}} \left[G(s, g(s), v_{\alpha(s,g(s))}) - G(t^{(r)}, g(t^{(r)}), v_{\alpha(t^{(r)}, g(t^{(r)}))}) \right] ds.$$

It follows from the regularity of v, α, f, σ and from the condition $\sigma(t,0)=0$ that there is $\gamma_0\in\Theta$ such that

$$\left|\Gamma_h^{(r)}\right| \leqslant h_0\gamma_0(h) \quad \text{for} \quad 0\leqslant r\leqslant K-1. \tag{3.8}$$

Then from (3.7) and from Assumption $H[f,G,\varphi]$ we conclude that

$$|v(t^{(r+1)},g(t^{(r+1)}))-z_h(t^{(r+1)},g_h(t^{(r+1)}))|\leqslant$$

$$[|v-z_h|]_{h.E}^{(r)}+h_0\sigma(t^{(r)},\varepsilon_{h.1}^{(r)}+\|v_{\alpha(t^{(r)},g(t^{(r)}))}-(\widehat{T}_hz_h)_{\alpha(t^{(r)},g_h(t^{(r)}))}\|_D)+h_0\gamma_0(h). \tag{3.9}$$

We have

$$\|v_{\alpha(t^{(r)},g(t^{(r)}))}-(\widehat{T}_hz_h)_{\alpha(t^{(r)},g_h(t^{(r)}))}\|_D\leqslant A^{(r)}(h)+B^{(r)}(h)$$

where

$$A^{(r)}(h)=\|v_{\alpha(t^{(r)},g(t^{(r)}))}-v_{\alpha(t^{(r)},g_h(t^{(r)}))}\|_D,\quad B^{(r)}(h)=\|(v-\widehat{T}_hz_h)_{\alpha(t^{(r)},g_h(t^{(r)}))}\|_D.$$

Let $\tilde{c}\in\mathbb{R}_+$ be such a constant that $|\partial_tv(t,x)|$, $\|\partial_xv(t,x)\|\leqslant\tilde{c}$ for $(t,x)\in\Omega$. Then

$$A^{(r)}(h)\leqslant\tilde{c}p\varepsilon_{h.1}^{(r)}.$$

Since

$$B^{(r)}(h)\leqslant\|v-\widehat{T}_hz_h\|_{\Omega^{(r)}},$$

it follows from Assumption $H[\widehat{T}_h]$ that there are β and C_1, $C_2\in\mathbb{R}_+$ such that

$$B^{(r)}(h)\leqslant\varepsilon_{h.0}^{(r)}+C_1\varepsilon_{h.1}^{(r)}+C_2\varepsilon_{h.2}^{(r)}+\beta(h)\quad\text{for}\quad 0\leqslant r\leqslant K-1.$$

We conclude from (3.9) and from the monotonicity of σ that for $0\leqslant r\leqslant K-1$

$$|v(t^{(r+1)},g(t^{(r+1)}))-z_h(t^{(r+1)},g_h(t^{(r+1)}))|\leqslant\varepsilon_{h.0}^{(r)}+h_0\sigma\left(t^{(r)},\bar{c}\varepsilon_h^{(r)}+\beta(h)\right)+h_0\gamma_0(h),$$

and, consequently,

$$\varepsilon_{h.0}^{(r+1)}\leqslant\varepsilon_{h.0}^{(r)}+h_0\sigma\left(t^{(r)},\bar{c}\varepsilon_h^{(r)}+\beta(h)\right)+h_0\gamma_0(h) \tag{3.10}$$

where $\bar{c}=2+\tilde{c}p+C_1+C_2$. In the same way we prove that there is $\gamma_1\in\Theta$ such that for $0\leqslant r\leqslant K-1$

$$\varepsilon_{h.1}^{(r+1)}\leqslant\varepsilon_{h.1}^{(r)}+h_0\sigma\left(t^{(r)},\bar{c}\varepsilon_h^{(r)}+\beta(h)\right)+h_0\gamma_1(h). \tag{3.11}$$

Let us now write a difference inequality for $\varepsilon_{h.2}$. From (2.4) it follows that

$$g_h(t^{(r+1)},t^{(s)},x^{(m)})-g_h(t^{(r+1)},t^{(s+\sigma_0)},x^{(m+\sigma')})=g_h(t^{(r)},t^{(s)},x^{(m)})-$$

$$g_h(t^{(r)},t^{(s+\sigma_0)},x^{(m+\sigma')})+h_0f(t^{(r)},g_h(t^{(r)},t^{(s)},x^{(m)}),(\widehat{T}_hz_h)_{\alpha(t^{(r)},g_h(t^{(r)},t^{(s)},x^{(m)}))})-$$

$$h_0f(t^{(r)},g_h(t^{(r)},t^{(s+\sigma_0)},x^{(m+\sigma')}),(\widehat{T}_hz_h)_{\alpha(t^{(r)},g_h(t^{(r)},t^{(s+\sigma_0)},x^{(m+\sigma')}))}). \tag{3.12}$$

Then from (3.6) and from Assumption $H[f,G,\varphi]$ we conclude that

$$\|g_h(t^{(r+1)},t^{(s)},x^{(m)})-g_h(t^{(r+1)},t^{(s+\sigma_0)},x^{(m+\sigma')})\|\leqslant$$

$$\varepsilon_{h.2}^{(r)}+h_0\sigma(t^{(r)},\varepsilon_{h.2}^{(r)}+\|(\widehat{T}_hz_h)_{\alpha(t^{(r)},g_h(t^{(r)},t^{(s)},x^{(m)}))}-(\widehat{T}_hz_h)_{\alpha(t^{(r)},g_h(t^{(r)},t^{(s+\sigma_0)},x^{(m+\sigma')})}\|_D). \tag{3.13}$$

A procedure, similar to that used to obtain (3.10), leads to

$$\varepsilon_{h.2}^{(r+1)} \leqslant \varepsilon_{h.2}^{(r)} + h_0\sigma(t^{(r)}, \hat{c}\varepsilon_h^{(r)} + \beta(h)), \quad 0 \leqslant r \leqslant K-1, \tag{3.14}$$

where

$$\hat{c} = 3 + 2C_1 + 2C_2 + \tilde{c}p. \tag{3.15}$$

Now, from (3.10), (3.11), (3.14), follows the difference inequality

$$\varepsilon_h^{(r+1)} \leqslant \varepsilon_h^{(r)} + h_0\sigma(t^{(r)}, \rho\varepsilon_h^{(r)} + \beta(h)) + h_0\gamma(h), \quad 0 \leqslant r \leqslant K-1, \tag{3.16}$$

where $\rho = \max\{\bar{c}, \hat{c}\} = \hat{c}$ and

$$\gamma = \max\{\gamma_0, \gamma_1\}. \tag{3.17}$$

Moreover, the initial estimate $\varepsilon_h^{(0)} \leqslant \bar{\psi}_0(h)$ is satisfied, where

$$\bar{\psi}_0(h) = \max\{\psi_0(h), \|h'\|\}. \tag{3.18}$$

Consider the Cauchy problem

$$\zeta'(s) = \sigma(s, \rho\zeta(s) + \beta(h)) + \gamma(h), \quad \zeta(0) = \bar{\psi}_0(h). \tag{3.19}$$

It follows from Assumption $H[f, G, \varphi]$ that there is $\varepsilon_0 > 0$ such that for $\|h\| < \varepsilon_0$ there exists the maximal solution ω_h of (3.19) and ω_h is defined on $[0, a]$. Moreover, we have

$$\lim_{h\to 0} \omega_h(s) = 0 \quad \text{uniformly on} \quad [0, a].$$

The function ω_h satisfy the recurrent inequality

$$\omega_h^{(r+1)} \geqslant \omega_h^{(r)} + h_0\sigma(t^{(r)}, \rho\omega_h^{(r)} + \beta(h)) + h_0\gamma(h), \quad 0 \leqslant r \leqslant K-1. \tag{3.20}$$

It follows from (3.16), (3.20) that

$$\varepsilon_h^{(r)} \leqslant \omega_h^{(r)} \quad \text{for} \quad 0 \leqslant r \leqslant K$$

and, consequently,

$$\max\left\{[|g[v] - g_h|]_h^{(r)},\ [|v - z_h|]_h^{(r)},\ 2\delta\widehat{\Omega}_h^{(r)}\right\} \leqslant \omega_h(a), \quad 0 \leqslant r \leqslant K.$$

Then assertion (3.2) is satisfied with $\psi(h) = \omega_h(a)$ and this complete the proof of the theorem. □

Remark 3.1. If all assumptions of Theorem 3.1 are satisfied with $\sigma(t, s) = Ls$, for $(t, s) \in [0, a] \times \mathbb{R}_+$, where $L \in \mathbb{R}_+$, then we have the estimates

$$\varepsilon_h^{(r)} \leqslant \bar{\psi}_0(h)\exp[L\hat{c}a] + \frac{L\beta(h) + \gamma(h)}{L\hat{c}}(\exp[L\hat{c}a] - 1) \quad \text{for} \quad L > 0$$

and

$$\varepsilon_h^{(r)} \leqslant \bar{\psi}_0(h) + \gamma(h)a \quad \text{for} \quad L = 0,$$

for $0 \leqslant r \leqslant K$, where $\hat{c}$, γ, $\psi_0(h)$ are given by (3.15), (3.17), (3.18), respectively.

4. Approximation operators

We give examples of the operator $\widehat{T}_h$ satisfying Assumption $H[\widehat{T}_h]$. Our investigations start with a method of, roughly speaking, uniformization of the mesh. By the uniformity we mean that the j-th coordinate of a node is a multiplicity of h_j, $0 \leqslant j \leqslant n$; the mesh $\widehat{\Omega}_h$ is uniform only with respect to the 0-th (i.e., time) coordinate. Defining $E_h = \Omega \cap \mathbb{R}^{1+n}_h$ and $\Omega_h = E_{0.h} \cup E_h \cup \partial_0 E_h$, we get a uniform mesh on Ω. We wish to approximate the values of the function $z_h : \widehat{\Omega}_h \to \mathbb{R}$ at uniform nodes of Ω_h, to be able to use the well-known interpolation operator $T_h : \mathbf{F}(\Omega_h, \mathbb{R}) \to C(\Omega, \mathbb{R})$ (see [6, Chapter 5]).

Let $h \in H$ and let Q be a finite subset of $I_h \times [-c, c]$, with the property

$$Q \cap (\{t^{(r)}\} \times [-c, c]) \neq \varnothing \quad \text{for} \quad 0 \leqslant r \leqslant K. \tag{4.1}$$

Later on we take $Q = \widehat{\Omega}_h$. We will define the "uniformization" operator

$$\mathcal{U} : \mathbf{F}(Q, \mathbb{R}) \to \mathbf{F}(\Omega_h, \mathbb{R}).$$

For this purpose, some auxiliary notation will be needed. Let us define the families of intervals:

$$\mathcal{F} = \left\{ \{t\} \times [a, b) \ : \ (t, a), (t, b) \in \mathbb{R}^{1+n}_h \right\}$$

and

$$\mathcal{F}^{(r,m)} = \left\{ F \in \mathcal{F} \ : \ (t^{(r)}, x^{(m)}) \in \overline{F}, \quad F \cap Q \neq \varnothing \right\},$$

and a number

$$d^{(r,m)} = \min \left\{ \|b - a\| \ : \ \{t\} \times [a, b) \in \mathcal{F}^{(r,m)} \right\}.$$

Then we put

$$Q^{(r,m)} = Q \cap \Big(\bigcup_{F \in \mathcal{F}^{(r,m)}_{min}} F \Big),$$

where

$$\mathcal{F}^{(r,m)}_{min} = \left\{ \{t\} \times [a, b) \in \mathcal{F}^{(r,m)} \ : \ \|b - a\| = d^{(r,m)} \right\}.$$

Due to (4.1), thus defined $Q^{(r,m)}$ is nonempty.

Definition 4.1. *Take $(t^{(r)}, x^{(m)}) \in \Omega_h$ and $z \in \mathbf{F}(Q, \mathbb{R})$. We put*

$$(\mathcal{U}z)(t^{(r)}, x^{(m)}) = \sum_{\nu=1}^{\kappa} w_\nu z(P_\nu), \tag{4.2}$$

for some $\{P_1, \ldots, P_\kappa\} = P^{(r,m)} \subset Q^{(r,m)}$. Two cases are distinguished here.

I. Suppose that $(t^{(r)}, x^{(m)}) \in Q$. We put then $P^{(r,m)} = \{(t^{(r)}, x^{(m)})\}$ and $w_1 = 1$.

II. Otherwise, if $(t^{(r)}, x^{(m)}) \notin Q$, we choose any nonempty subset $P^{(r,m)}$ of $Q^{(r,m)}$ and any nonnegative w_ν such that

$$\sum_{\nu=1}^{\kappa} w_\nu = 1. \tag{4.3}$$

Example 4.1. A simple approximation may be done in the following way. In case II of the above Definition, we take an arbitrarily chosen one-element subset $P^{(r,m)} = \{(t^{(r)}, p)\}$ of $Q^{(r,m)}$ and we put $w_1 = 1$.

Example 4.2. A more complicated approximation may be constructed using Shepard's interpolation. In case II of the above Definition, we take $P^{(r,m)} = Q^{(r,m)}$ and we put

$$w_\nu = \|p_\nu - x^{(m)}\|^{-1}\left(\sum_{i=1}^{\kappa}\|p_i - x^{(m)}\|^{-1}\right)^{-1}, \quad 1 \leqslant \nu \leqslant \kappa,$$

where $P_\nu = (t^{(r)}, p_\nu)$, $1 \leqslant \nu \leqslant \kappa$.

We will prove that Definition 4.1 is suitable for our purposes, i.e. assures the fulfillment of Assumption $H[\widehat{T}_h]$.

Note that for $(t^{(r)}, p) \in Q^{(r,m)}$, we have $\|p - x^{(m)}\| \leqslant d^{(r,m)}$. If we take $Q = \widehat{\Omega}_h$, then

$$\|p - x^{(m)}\| \leqslant d^{(r,m)} \leqslant \delta\widehat{\Omega}_h^{(r)} + \|h'\|. \tag{4.4}$$

4.1. Interpolation operator T_h and approximation operator $\widehat{T}_h$. Now we describe the interpolation operator $T_h : \mathbf{F}(\Omega_h, \mathbb{R}) \to \mathbf{F}(\Omega, \mathbb{R})$ presented in [6].

Suppose that $z : \Omega_h \to \mathbb{R}$. For $(t, x) \in \Omega$ three cases will be distinguished.

I. Then there is $(r, m) \in \mathbb{Z}^{1+n}$ such that $(t^{(r)}, x^{(m)})$, $(t^{(r+1)}, x^{(m+1)}) \in \Omega_h$ and $t^{(r)} \leqslant t \leqslant t^{(r+1)}$, $x^{(m)} \leqslant x \leqslant x^{(m+1)}$, where $m + 1 = (m_1 + 1, \ldots, m_n + 1)$. We define

$$(T_h z)(t,x) = \left(1 - \frac{t - t^{(r)}}{h_0}\right)\sum_{\sigma' \in \Sigma} z^{(r,m+\sigma')}\left(\frac{x - x^{(m)}}{h'}\right)^{\sigma'}\left(1 - \frac{x - x^{(m)}}{h'}\right)^{1-\sigma'}$$

$$+\left(\frac{t - t^{(r)}}{h_0}\right)\sum_{\sigma' \in \Sigma} z^{(r+1,m+\sigma')}\left(\frac{x - x^{(m)}}{h'}\right)^{\sigma'}\left(1 - \frac{x - x^{(m)}}{h'}\right)^{1-\sigma'}$$

where

$$\left(\frac{x - x^{(m)}}{h'}\right)^{\sigma'} = \prod_{j=1}^{n}\left(\frac{x_j - x_j^{(m_j)}}{h_j}\right)^{\sigma_j},$$

and

$$\left(1 - \frac{x - x^{(m)}}{h'}\right)^{1-\sigma'} = \prod_{j=1}^{n}\left(1 - \frac{x_j - x_j^{(m_j)}}{h_j}\right)^{1-\sigma_j}$$

and we take $0^0 = 1$ in the above formulas.

II. Suppose that $|x_j| > \bar{N}_j h_j$ for some j, $1 \leqslant j \leqslant n$, where $\bar{N} = (\bar{N}_1, \ldots, \bar{N}_n) \in \mathbb{N}^n$ is defined by the relation $\bar{N} \diamond h' \leqslant c < (\bar{N} + 1) \diamond h'$. Then we put $T_h z(t, x) = T_h z(t, \tilde{x})$, where

$$\tilde{x}_j = \begin{cases} x_j, & |x_j| \leqslant \bar{N}_j h_j \\ -\bar{N}_j h_j, & x_j < -\bar{N}_j h_j \\ \bar{N}_j h_j, & x_j > \bar{N}_j h_j, \end{cases} \qquad 1 \leqslant j \leqslant n.$$

III. The last case is $Kh_0 < t \leqslant a$. Then we set $T_h z(t, x) = T_h z(Kh_0, x)$.

Note that T_h is a linear operator from $\mathbf{F}(\Omega_h, \mathbb{R})$ to $C(\Omega, \mathbb{R})$. For fixed r, $0 \leqslant r \leqslant K$, we define $\Omega_h^{(r)} = \Omega_h \cap \Omega^{(r)}$, and for $z_h : \Omega_h^{(r)} \to \mathbb{R}$ we put

$$\|z_h\|_{\Omega_h^{(r)}} = \max\left\{\, |z_h^{(i,m)}| \;:\; (t^{(i)}, x^{(m)}) \in \Omega_h^{(r)} \right\}.$$

It is easy to see that the estimate

$$\|T_h z_h\|_{\Omega^{(r)}} = \|z_h\|_{\Omega_h^{(r)}} \tag{4.5}$$

holds for $-K_0 \leqslant r \leqslant K$.

Let $Q = \widehat{\Omega}_h$ in Definition 4.1. Then we define

$$\widehat{T}_h w = T_h \mathcal{U} w \tag{4.6}$$

for $w \in \mathbf{F}(\widehat{\Omega}_h, \mathbb{R})$.

4.2. Properties of the operator $\mathcal{U}$ and approximation operators. To prove that the the above-defined operator $\widehat{T}_h$ satisfies Assumption $H[\widehat{T}_h]$, we introduce an auxiliary mesh $\widetilde{\Omega}_h$. Suppose that $v \in C^1(\Omega, \mathbb{R})$ and consider the set of bicharacteristics $g[v](\cdot, t, x)$, $(t, x) \in S_h$. Then we write

$$\widetilde{E}_h = \{(t^{(r)}, g[v](t^{(r)}, t, x)) \in E : (t, x) \in S_h, \quad 0 \leqslant r \leqslant K\}$$

and

$$\widetilde{\Omega}_h = \widetilde{E}_h \cup E_{0.h} \cup \partial_0 E_h, \qquad \widetilde{\Omega}_h^{(r)} = \widetilde{\Omega}_h \cap \Omega^{(r)}, \quad 0 \leqslant r \leqslant K.$$

Moreover, with $Q = \widetilde{\Omega}_h$ taken in Definition 4.1, we put

$$\widetilde{T}_h w = T_h \mathcal{U} w \tag{4.7}$$

for $w \in \mathbf{F}(\widetilde{\Omega}_h, \mathbb{R})$. We define $\delta\widetilde{\Omega}_h$ analogously to $\delta\widehat{\Omega}_h$, by putting g instead of g_h in the definition of the latter. Similarly to (4.4), we have

$$\|p - x^{(m)}\| \leqslant d^{(r,m)} \leqslant \delta\widetilde{\Omega}_h^{(r)} + \|h'\| \tag{4.8}$$

for $(t^{(r)}, p) \in Q^{(r,m)}$.

We will use the restriction operators $R_h : \mathbf{F}(\Omega, \mathbb{R}) \to \mathbf{F}(\Omega_h, \mathbb{R})$, $\widehat{R}_h : \mathbf{F}(\Omega, \mathbb{R}) \to \mathbf{F}(\widehat{\Omega}_h, \mathbb{R})$ and $\widetilde{R}_h : \mathbf{F}(\Omega, \mathbb{R}) \to \mathbf{F}(\widetilde{\Omega}_h, \mathbb{R})$, defined by

$$R_h v = v|_{\Omega_h}, \qquad \widehat{R}_h v = v|_{\widehat{\Omega}_h} \quad \text{and} \quad \widetilde{R}_h v = v|_{\widetilde{\Omega}_h}$$

for $v \in \mathbf{F}(\Omega, \mathbb{R})$.

Lemma 4.1. *Suppose that $0 \leqslant r \leqslant K$, $h \in H$, and*

1) $v \in C^1(\Omega, \mathbb{R})$, $\tilde{c} \in \mathbb{R}_+$ are such that $|\partial_t v(t, x)|$, $\|\partial_x v(t, x)\| \leqslant \tilde{c}$ on Ω;

2) (g_h, z_h) is a solution of the difference system (2.3)–(2.5),

3) $g[v](\cdot, t, x)$ is the set of bicharacteristics for (1.1).

Then:

(i) $\|\mathcal{U}\widetilde{R}_h v - R_h v\|_{\Omega_h^{(r)}} \leqslant \tilde{c}\left(\delta\widetilde{\Omega}_h^{(r)} + \|h'\|\right)$,

(ii) $\|T_h R_h v - v\|_{\Omega^{(r)}} \leqslant \tilde{c}\|h\|$,

(iii) $\|\widetilde{T}_h \widetilde{R}_h v - v\|_{\Omega^{(r)}} \leqslant \tilde{c}\left(\delta\widetilde{\Omega}_h^{(r)} + 2\|h\|\right)$,

(iv) $\|\widetilde{T}_h \widetilde{R}_h v - \widehat{T}_h z_h\|_{\Omega^{(r)}} \leqslant [|v - z_h|]_h^{(r)} + \tilde{c}\,[|g[v] - g_h|]_h^{(r)} + \tilde{c}\,\delta\widehat{\Omega}_h^{(r)} + \tilde{c}\gamma(h)$,

where

$$\gamma(h) = \delta\widetilde{\Omega}_h^{(r)} + 2\|h\|. \tag{4.9}$$

Proof. ***(i)*** Let us take $Q = \widetilde{\Omega}_h$ in Definition 4.1. We should estimate the difference

$$\left|(\mathcal{U}\widetilde{R}_h v - R_h v)(t^{(i)}, x^{(m)})\right|$$

for $(t^{(i)}, x^{(m)}) \in \Omega^{(r)}$. In case I from that Definition, i.e., when $(t^{(i)}, x^{(m)}) \in \widetilde{\Omega}_h$, the above difference vanishes. In case II, the assertion follows from relations (4.3), (4.4) and from the mean value theorem.

(ii) We omit the proof of this part, which is similar to the proof of Lemma 5.27 in [6].

(iii) The estimate follows from the triangle inequality, involving (i), with relation (4.5) applied to its left-hand side, and (ii).

(iv) Since from (4.6), (4.7) we have $\widetilde{T}_h\widetilde{R}_h v - \widehat{T}_h z_h = T_h(\mathcal{U}\widetilde{R}_h v - \mathcal{U}z_h)$, relation (4.5) implies

$$\|\widetilde{T}_h\widetilde{R}_h v - \widehat{T}_h z_h\|_{\Omega^{(r)}} = \|\mathcal{U}\widetilde{R}_h v - \mathcal{U}z_h\|_{\Omega_h^{(r)}}.$$

Let $(t^{(i)}, x^{(m)}) \in \Omega_h^{(r)}$ be such that

$$\|\mathcal{U}\widetilde{R}_h v - \mathcal{U}z_h\|_{\Omega_h^{(r)}} = \left|(\mathcal{U}\widetilde{R}_h v - \mathcal{U}z_h)(t^{(i)}, x^{(m)})\right|.$$

Two possibilities can occur, either (a) $(t^{(i)}, x^{(m)}) \in E_{h.0} \cup \partial_0 E_h$, or (b) $(t^{(i)}, x^{(m)}) \notin E_{h.0} \cup \partial_0 E_h$.

Consider the case (a). Then $(t^{(i)}, x^{(m)}) \in \widehat{\Omega}_h \cap \Omega_h$ and $(t^{(i)}, x^{(m)}) \in \widetilde{\Omega}_h \cap \Omega_h$, and twice the case I from Definition 4.1 holds, namely

$$\mathcal{U}\widetilde{R}_h v(t^{(i)}, x^{(m)}) = \widetilde{R}_h v(t^{(i)}, x^{(m)})$$

and

$$\mathcal{U}z_h(t^{(i)}, x^{(m)}) = z_h(t^{(i)}, x^{(m)}).$$

Moreover, $\widetilde{R}_h v(t^{(i)}, x^{(m)}) = v(t^{(i)}, x^{(m)})$, and hence

$$\left|(\mathcal{U}\widetilde{R}_h v - \mathcal{U}z_h)(t^{(i)}, x^{(m)})\right| = |v(t^{(i)}, x^{(m)}) - z_h(t^{(i)}, x^{(m)})| \leqslant [|v - z_h|]_{h.\partial}^{(r)} \leqslant [|v - z_h|]_h^{(r)}.$$

Then (iv) is proved in the case (a). Consider now the case (b). From Definition 4.1 it follows that

$$\mathcal{U}z_h(t^{(i)}, x^{(m)}) = \sum_{\nu=1}^{\kappa} w_\nu z_h(P_\nu) \tag{4.10}$$

for some $P_\nu \in \widehat{\Omega}_h$, $w_\nu > 0$. We have $x^{(m)} \in [-b, b]$, hence $P_\nu \in \{t^{(i)}\} \times [-b, b]$, and

$$P_\nu = (t^{(i)}, g_h(t^{(i)}, \alpha_\nu)) \quad \text{for some} \quad \alpha_\nu \in S_h, \quad 1 \leqslant \nu \leqslant \kappa.$$

Then we define the set $\{\bar{P}_\nu\}_{\nu=1}^{\kappa} \subset \widetilde{\Omega}_h$ by

$$\bar{P}_\nu = (t^{(i)}, g[v](t^{(i)}, \alpha_\nu)), \quad 1 \leqslant \nu \leqslant \kappa.$$

Then

$$|v(\bar{P}_\nu) - v(P_\nu)| \leqslant \tilde{c}\, \|g[v](t^{(i)}, \alpha_\nu) - g_h(t^{(i)}, \alpha_\nu)\| \leqslant \tilde{c}\, [|g[v] - g_h|]_h^{(r)}. \tag{4.11}$$

Moreover, we have

$$|v(\bar{P}_\nu) - z_h(P_\nu)| = |v(t^{(i)}, g[v](t^{(i)}, \alpha_\nu)) - z_h(t^{(i)}, g_h(t^{(i)}, \alpha_\nu))| \leqslant [|v - z_h|]_{h.E}^{(r)} \leqslant [|v - z_h|]_h^{(r)}. \tag{4.12}$$

We put

$$A = \sum_{\nu=1}^{\kappa} w_\nu v(P_\nu) \quad \text{and} \quad \bar{A} = \sum_{\nu=1}^{\kappa} w_\nu v(\bar{P}_\nu).$$

Due to the continuity of $v(t^{(i)}, \cdot)$ and relation (4.3), the iterative use of the intermediate value theorem gives us the existence of a point $y \in \operatorname{conv}\{g_h(t^{(i)}, \alpha_\nu)\}_{\nu=1}^{\kappa} \subset \mathbb{R}^n$ such that

$$v(t^{(i)}, y) = A.$$

Similarly, there exists a point $\tilde{y} \in \mathbb{R}^n$ such that

$$v(t^{(i)}, \tilde{y}) = \mathcal{U}\widetilde{R}_h v(t^{(i)}, x^{(m)}).$$

From the relation $y \in \operatorname{conv}\{g_h(t^{(i)}, \alpha_\nu)\}_{\nu=1}^{\kappa}$ and from (4.4)

$$\|y - x^{(m)}\| \leqslant \max_{1 \leqslant \nu \leqslant \kappa} \|g_h(t^{(i)}, \alpha_\nu) - x^{(m)}\| \leqslant \delta\widehat{\Omega}_h^{(r)} + \|h'\|.$$

Let $Q^{(i,m)}$ be a set from Definition 4.1 with $Q = \widetilde{\Omega}_h$. Since $(t^{(i)}, \tilde{y}) \in \operatorname{conv} Q^{(i,m)}$, we have from (4.8)

$$\|\tilde{y} - x^{(m)}\| \leqslant \max\left\{ \|p - x^{(m)}\| \ : \ (t^{(i)}, p) \in Q^{(i,m)} \right\} \leqslant \delta\widetilde{\Omega}_h^{(r)} + \|h'\|$$

Then

$$\|\tilde{y} - y\| \leqslant \|\tilde{y} - x^{(m)}\| + \|x^{(m)} - y\| \leqslant \delta\widehat{\Omega}_h^{(r)} + \gamma(h)$$

and, consequently,

$$|v(t^{(i)}, \tilde{y}) - v(t^{(i)}, y)| \leqslant \tilde{c}\delta\widehat{\Omega}_h^{(r)} + \tilde{c}\gamma(h). \tag{4.13}$$

Now, from (4.11)–(4.13), we have

$$|\,\mathcal{U}\widetilde{R}_h v(t^{(i)}, x^{(m)}) - \mathcal{U}z_h(t^{(i)}, x^{(m)})| \leqslant |\,\mathcal{U}\widetilde{R}_h v(t^{(i)}, x^{(m)}) - A| + |A - \bar{A}| + |\bar{A} - \mathcal{U}z_h(t^{(i)}, x^{(m)})| \leqslant$$

$$|v(t^{(i)}, \tilde{y}) - v(t^{(i)}, y)| + \sum_{\nu=1}^{\kappa} w_\nu |v(t^{(i)}, P_\nu) - v(t^{(i)}, \bar{P}_\nu)| + \sum_{\nu=1}^{\kappa} w_\nu |v(t^{(i)}, \bar{P}_\nu) - z_h(t^{(i)}, P_\nu)| \leqslant$$

$$\tilde{c}\delta\widehat{\Omega}_h^{(r)} + \tilde{c}\gamma(h) + \sum_{\nu=1}^{\kappa} w_\nu \left(\tilde{c}[|g[v] - g_h|]_h^{(r)} + [|v - z_h|]_h^{(r)} \right) =$$

$$\tilde{c}\delta\widehat{\Omega}_h^{(r)} + \tilde{c}\gamma(h) + \tilde{c}[|g[v] - g_h|]_h^{(r)} + [|v - z_h|]_h^{(r)}.$$

Then (iv) is proved also in the case (b). This completes the proof of Lemma 4.1. □

Remark 4.1. Note that from the classical theorems on the continuous dependence of solutions of Cauchy problems on the initial data

$$\lim_{h \to 0} \delta\widetilde{\Omega}_h^{(r)} = 0 \quad \text{for} \quad 0 \leqslant r \leqslant K$$

follows. Then from Lemma 4.1, (iii) and (iv) it follows that condition 3) from Assumption $H[\widehat{T}_h]$ is fulfilled by the approximation operator $\widehat{T}_h$ defined by (4.6).

5. Error estimate

Lemma 5.1. *Suppose that all assumptions of Theorem* 3.1 *are satisfied with* $\sigma(t,s) = Ls$, *for* $(t,s) \in [0,a] \times \mathbb{R}_+$, *where* $L \in \mathbb{R}_+$, *and*

1) $\tilde{c} \in \mathbb{R}_+$ *is such that* $|\partial_t v(t,x)|$, $\|\partial_x v(t,x)\| \leqslant \tilde{c}$ *on* Ω;

2) $A \in \mathbb{R}_+$ *is such that*

$$\|f(t,x,v_{\alpha(t,x)})\| \leqslant A \quad on \quad \Omega;$$

3) *the estimates*

$$|f(t,x,w) - f(\bar{t},x,w)| \leqslant L|t-\bar{t}|, \qquad \|G(t,x,w) - G(\bar{t},x,w)\| \leqslant L|t-\bar{t}|$$

hold on Ξ;

4) *the above-defined* (*see Definition* 4.1 *and* (4.6)) *approximation operator* $\widehat{T}_h$ *is used in the numerical method* (2.3)–(2.5).

Then (3.2) *holds with*

$$\psi(h) = \max\{\psi_0(h), \|h'\|\} \exp[L\hat{c}a] + \frac{L\beta(h) + \gamma(h)}{L\hat{c}}(\exp[L\hat{c}a] - 1) \quad for \quad L > 0$$

and

$$\psi(h) = \max\{\psi_0(h), \|h'\|\} + \gamma(h)a \quad for \quad L = 0,$$

where $\beta(h) = \tilde{c}(L^* + 4)\|h\|$, $\hat{c} = 3 + 5\tilde{c} + \tilde{c}p$, $\gamma(h) = h_0 L(1+A)(1+\tilde{c}p)/2$, *and* $L^* = \max\{1, A\} \exp[L(1+\tilde{c}p)a]$.

Proof. If we take the prescribed approximation operators, Lemma 4.1 assures the fulfillment of Assumption $H[\widehat{T}_h]$ with $C_1 = C_2 = \tilde{c}$, hence $\hat{c}$.

Condition 2) implies

$$\|g[v](s,t,x) - g[v](t^{(r)},t,x)\| \leqslant A|s - t^{(r)}|,$$

which, together with the global Lipschitz condition for f and G with constant L, gives

$$\|G(s, g[v](s,t,x), v_{\alpha(s,g[v](s,t,x))}) - G(t^{(r)}, g[v](t^{(r)},t,x), v_{\alpha(t^{(r)},g[v](t^{(r)},t,x))})\| \leqslant$$
$$L|s - t^{(r)}|(1+A)(1+\tilde{c}p)$$

and a similar estimate for f. Recalling the definition of γ_0 and γ_1 from the proof of Theorem 3.1 and integrating the above inequality over $[t^{(r)}, t^{(r+1)}]$, we get

$$\gamma_0(h) = \gamma_1(h) = \frac{1}{2} h_0 L(1+A)(1+\tilde{c}p),$$

which is also the formula for $\gamma(h)$.

Again, Lemma 4.1, together with the definition of β and $\tilde{\gamma}$, gives

$$\beta(h) = 2\tilde{c}\left(\delta\widetilde{\Omega}_h^{(K)} + 2\|h\|\right).$$

Moreover, from the classical theorem on the continuous dependence on the initial data and from the definition of $\delta\widetilde{\Omega}_h$ it follows that $\delta\widetilde{\Omega}_h^{(r)} \leqslant L^*\|h\|/2$ for all r, $0 \leqslant r \leqslant K$. This implies

$$\beta(h) = 2\tilde{c}(L^*/2 + 2)\|h\|.$$

Putting the above inequalities together and using the formulas from Remark 3.1, we get the claimed error estimate. ∎

6. Numerical examples

Example 6.1. Let $n = 2$, $a \leqslant 1$ and $E = [0,a] \times [-1,1]^2$, $E_0 = \{0\} \times [-1,1]^2$, $\partial_0 E = [0,a] \times ([-1,1]^2 \setminus (-1,1)^2)$. Consider the mixed problem

$$z_t(t,x) + [-2x_1 - q(x_2)]z_{x_1}(t,x) + [-2x_2 + q(z(t,tx_1,tx_2))]z_{x_2}(t,x) =$$

$$\exp(t(x_1^2 - x_2^2))[(x_1^2 - x_2^2)(1-4t) + 2t(x_2 q(z(t,tx_1,tx_2)) - x_1 q(x_2))], \tag{6.1}$$

$$z(t,x,y) = \begin{cases} 1 & \text{on } E_0, \\ e^{t(1-x_2^2)} & \text{on } \partial_0 E \cap \{|x_1| = 1\}, \\ e^{-t(1-x_1^2)} & \text{on } \partial_0 E \cap \{|x_2| = 1\}. \end{cases} \tag{6.2}$$

where

$$q(x) = \frac{1}{p}\, g\left(\frac{x}{4}\right), \quad g(x) = \begin{cases} -p, & x \leqslant -p, \\ -x\ln(|x|), & |x| < p, \quad x \neq 0, \\ 0, & x = 0, \\ p, & x \geqslant p, \end{cases} \tag{6.3}$$

with $p = \exp(-1)$. Note that g, and hence q, is continuous and non-Lipschitz; nevertheless, the conditions of the convergence theorem are fulfilled, since for $\sigma(t,p) = a_1 p + a_2 p|\ln p|$, a_1, $a_2 > 0$, and for $\rho \geqslant 1$ the only solution of the Cauchy problem (3.1) with is $\zeta(t) = 0$, $t \in [0,a]$. The solution of the problem is given by $\tilde{z}(t,x) = e^{t(x_1^2 - x_2^2)}$. For problem (6.1), (6.2) we use the numerical method of bicharacteristics (2.3)–(2.5), involving approximation operator $\widehat{T}_h = T_h \mathcal{U}$, with $\mathcal{U}$ from Example 4.1. Denote by (g_h, z_h) the solution of this method. For fixed $t^{(r)}$, $0 \leqslant r \leqslant K$, we put

$$\varepsilon^{(r)}_{h.\max} = \max\{|(z_h - v)(t^{(r)}, y)| : (t^{(r)}, y) \in \widehat{\Omega}_h\},$$

$$\varepsilon^{(r)}_{h.\mathrm{mean}} = \left(\#\{y : (t^{(r)}, y) \in \widehat{\Omega}_h\}\right)^{-1} \sum_{(t^{(r)}, y) \in \widehat{\Omega}_h} |(z_h - v)(t^{(r)}, y)|.$$

Consider now the Lax scheme for (6.1), (6.2). We denote the solution of this classical method by $\bar{z}_h$. Then we define the errors $\bar{\varepsilon}^{(r)}_{h.\max}$, $\bar{\varepsilon}^{(r)}_{h.\mathrm{mean}}$ in a similar way as above, with $\bar{z}_h$ instead of z_h and with Ω_h instead of $\widehat{\Omega}_h$.

We put $h_0 = 10^{-4}$, $h_1 = h_2 = 2 \cdot 10^{-2}$, $a = 0.25$ and obtain the following experimental values of the above-defined errors.

$t^{(r)}$	$\bar{\varepsilon}^{(r)}_{h.\max}$	$\varepsilon^{(r)}_{h.\max}$	$\bar{\varepsilon}^{(r)}_{h.\mathrm{mean}}$	$\varepsilon^{(r)}_{h.\mathrm{mean}}$
0.0625	$2.548768 \cdot 10^{-4}$	$3.157028 \cdot 10^{-5}$	$1.287677 \cdot 10^{-4}$	$8.305750 \cdot 10^{-6}$
0.1250	$1.514947 \cdot 10^{-3}$	$5.104396 \cdot 10^{-5}$	$8.485705 \cdot 10^{-4}$	$1.453428 \cdot 10^{-5}$
0.1875	$4.166585 \cdot 10^{-3}$	$6.373904 \cdot 10^{-5}$	$2.473140 \cdot 10^{-3}$	$1.918148 \cdot 10^{-5}$
0.2500	$8.442145 \cdot 10^{-3}$	$7.185911 \cdot 10^{-5}$	$5.190896 \cdot 10^{-3}$	$2.258027 \cdot 10^{-5}$

The approximate computing time for the classical method on a PC, with the AMD Duron™, 1.4 GHz CPU, cache of size 64 kB and 256 MB RAM, is 31 s compared to 1750 s for the numerical method of bicharacteristics.

For the same domain and another set of steps of the mesh, $h_0 = 5\cdot 10^{-3}$, $h_1 = h_2 = 2\cdot 10^{-3}$, the errors of the classical Lax scheme become greater than 10^6 for $r \geqslant 19$. The errors of our method are given below.

$t^{(r)}$	$\varepsilon^{(r)}_{h.\max}$	$\varepsilon^{(r)}_{h.\mathrm{mean}}$
0.0625	$1.544952 \cdot 10^{-3}$	$4.181424 \cdot 10^{-4}$
0.1250	$2.509618 \cdot 10^{-3}$	$7.121527 \cdot 10^{-4}$
0.1875	$3.146780 \cdot 10^{-3}$	$9.196634 \cdot 10^{-4}$
0.2500	$3.563794 \cdot 10^{-3}$	$1.065446 \cdot 10^{-3}$

The approximate computing time for the numerical method of bicharacteristics, on the same machine, is 85 s.

Example 6.2. We set $n = 1$, $\tau = 1$, and

$$E = [0, a] \times [-1, 1], \quad E_0 = \{0\} \times [-1.5, 1.5], \quad \partial_0 E = ([0, a] \times [-1.5, 1.5]) \setminus E.$$

Consider the mixed problem

$$\partial_t u(t,x) + q\Big(\int_{\beta(x)}^{\gamma(x)} u(t,s)\,ds\Big)(1 + \partial_x u(t,x)) - 2x\partial_x u(t,x) = F(t,x), \tag{6.4}$$

$$u(t,x) = \frac{1}{2}t(1 + 2t + x^4) \qquad \text{for} \qquad (t,x) \in E_0 \cup \partial_0 E, \tag{6.5}$$

where

$$\beta(x) = \frac{5}{4}x - \frac{1}{4}, \quad \gamma(x) = \frac{5}{4}x + \frac{1}{4}, \quad F(t,x) = 2t + x^2(1 - 4t) + (1 + 2tx)\, q(\tilde{g}(t,x))$$

and

$$\tilde{g}(t,x) = \begin{cases} t\left[\dfrac{625}{2048}\left(-x^5 + x^4 + \dfrac{26}{15}x^3 + \dfrac{34}{25}x^2 - \dfrac{9}{5}x\right) + \dfrac{1}{2}t - \dfrac{4189}{30720}\right], & x \in [-1, -0.6], \\ t\left(\dfrac{1}{2}t + \dfrac{25}{32}x^2 + \dfrac{1}{96}\right), & x \in (-0.6, 0.6), \\ t\left[\dfrac{625}{2048}\left(x^5 + x^4 - \dfrac{26}{15}x^3 + \dfrac{34}{25}x^2 + \dfrac{9}{5}x\right) + \dfrac{1}{2}t - \dfrac{4189}{30720}\right], & x \in [0.6, 1]. \end{cases}$$

The solution of the problem is given by $\tilde{v}(t,x) = t(t + x^2)$.

We compare the method to the classical Lax scheme and define the errors of both methods in the same way as in the previous Example. Moreover, we use the numerical method of bicharacteristics involving the Shepard interpolation (i.e., we implement the approximation operator $\widehat{T}_h$ with the aid of $\mathcal{U}$ from Example 4.2), and denote its errors by $\tilde{\varepsilon}^{(r)}_{h.\max}$ and $\tilde{\varepsilon}^{(r)}_{h.\mathrm{mean}}$.

Taking $h_0 = 10^{-5}$, $h_1 = 2 \cdot 10^{-3}$, $a = 0.25$, we obtain the following experimental values of the errors.

$t^{(r)}$	$\bar{\varepsilon}^{(r)}_{h.\max}$	$\varepsilon^{(r)}_{h.\max}$	$\tilde{\varepsilon}^{(r)}_{h.\max}$	$\bar{\varepsilon}^{(r)}_{h.\mathrm{mean}}$	$\varepsilon^{(r)}_{h.\mathrm{mean}}$	$\tilde{\varepsilon}^{(r)}_{h.\mathrm{mean}}$
0.0625	$7.598918 \cdot 10^{-4}$	$5.159632 \cdot 10^{-6}$	$1.557761 \cdot 10^{-6}$	$6.896522 \cdot 10^{-4}$	$1.203477 \cdot 10^{-6}$	$7.579235 \cdot 10^{-7}$
0.1250	$2.959015 \cdot 10^{-3}$	$1.728652 \cdot 10^{-5}$	$2.505059 \cdot 10^{-6}$	$2.564387 \cdot 10^{-3}$	$3.323471 \cdot 10^{-6}$	$1.265340 \cdot 10^{-6}$
0.1875	$6.503220 \cdot 10^{-3}$	$3.428650 \cdot 10^{-5}$	$2.955749 \cdot 10^{-6}$	$5.422428 \cdot 10^{-3}$	$6.699800 \cdot 10^{-6}$	$1.552376 \cdot 10^{-6}$
0.2500	$1.132334 \cdot 10^{-2}$	$5.437411 \cdot 10^{-5}$	$3.030813 \cdot 10^{-6}$	$9.120413 \cdot 10^{-3}$	$1.141974 \cdot 10^{-5}$	$1.661398 \cdot 10^{-6}$

The approximate computing time for the classical method on the Intel®Xeon™CPU 3.06 GHz machine cache of size 512 kB and 4 GB RAM, is 14 s compared to 440 s for the numerical method of bicharacteristics with the single-node approximation and 670 s for the Shepard variant.

For the same domain and another set of steps of the mesh, $h_0 = h_1 = 5 \cdot 10^{-4}$, the errors of the classical Lax scheme become greater than 10^6 for $r \geqslant 53$. The errors of our method, in both variants, are given below.

$t^{(r)}$	$\varepsilon^{(r)}_{h.\max}$	$\tilde{\varepsilon}^{(r)}_{h.\max}$	$\varepsilon^{(r)}_{h.\mathrm{mean}}$	$\tilde{\varepsilon}^{(r)}_{h.\mathrm{mean}}$
0.0625	$7.956374 \cdot 10^{-5}$	$7.951170 \cdot 10^{-5}$	$2.673792 \cdot 10^{-5}$	$2.672484 \cdot 10^{-5}$
0.1250	$1.300688 \cdot 10^{-4}$	$1.297184 \cdot 10^{-4}$	$4.680133 \cdot 10^{-5}$	$4.674174 \cdot 10^{-5}$
0.1875	$1.564073 \cdot 10^{-4}$	$1.553630 \cdot 10^{-4}$	$6.244022 \cdot 10^{-5}$	$6.233182 \cdot 10^{-5}$
0.2500	$1.643924 \cdot 10^{-4}$	$1.621890 \cdot 10^{-4}$	$7.500613 \cdot 10^{-5}$	$7.485633 \cdot 10^{-5}$

The approximate computing time for the numerical method of bicharacteristics, on the same machine, is 1 s.

Remark 6.1. The huge values of errors of the classical Lax scheme in the second experiment from Example 6.1 and in the second experiment from Example 6.2 are due to the fact that the steps of the mesh don't satisfy the CFL condition (1.3).

The method described in the present paper has a potential for applications in the numerical solution of first order nonlinear differential equations with deviated variables and first order integral differential equations.

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