# Generating Functions' Examples

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## 1 Introduction

**Definition 1.1.** An *ordinary* generating function $A(x)$ of $a(n)$ is the formal power series

$$A(x) = \sum_{n\geq 0} a(n)x^n, \tag{1.1}$$

while the *exponential* generating function $B(x)$ of $b(n)$ is

$$B(x) = \sum_{n\geq 0} b(n)\frac{x^n}{n!}. \tag{1.2}$$

Let us define *Fibonomial* generating function as

$$F(x) = \sum_{n\geq 0} f(n)\frac{x^n}{F_n!}, \tag{1.3}$$

where $F_0 = 1, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ is the sequence of Fibonacci numbers.

Let us recall *convolution* for ordinary generating functions

$$\left(\sum_{n\geq 0} a_n x^n\right)\left(\sum_{n\geq 0} b_n x^n\right) = \sum_{n\geq 0}\left(\sum_{k=0}^{n} a_n b_{n-k}\right)x^n, \tag{1.4}$$

for exponential generating functions

$$\left(\sum_{n\geq 0} \frac{a_n x^n}{n!}\right)\left(\sum_{n\geq 0} \frac{b_n x^n}{n!}\right) = \sum_{n\geq 0}\left(\sum_{k=0}^{n} \binom{n}{k} a_n b_{n-k}\right)\frac{x^n}{n!}, \tag{1.5}$$

and for Fibonomial generating functions

$$\left(\sum_{n\geq 0} \frac{a_n x^n}{F_n!}\right)\left(\sum_{n\geq 0} \frac{b_n x^n}{F_n!}\right) = \sum_{n\geq 0}\left(\sum_{k=0}^{n} \binom{n}{k}_F a_n b_{n-k}\right)\frac{x^n}{F_n!}. \tag{1.6}$$

**Definition 1.2.** If $F(x) \in \mathbb{C}[[x]]$ satisfies $F(0) = 0$, then we can define for any $\lambda \in \mathbb{C}$ the formal power series

$$(1 + F(x))^\lambda = \sum_{n\geq 0} \binom{\lambda}{n} F(x)^n. \tag{1.7}$$

**Example 1.1.** From [1]. Let $\mu(n)$ be the Möbius function from the number theory; that is, $\mu(1) = 1$, $\mu(n) = 0$ if $n$ is divisible by the square of an integer greather than one, and $\mu(n) = (-1)^r$ if $n$ is the product of $r$ distinct primes. Find a simple expression for the power series

$$F(x) = \prod_{n\geq 1}(1-x^n)^{-\mu(n)/n}. \tag{1.8}$$

We apply log to $F(x)$ to get

$$\begin{aligned}\log F(x) &= \log \prod_{n\geq 1}(1-x^n)^{-\mu(n)/n}\\ &= \sum_{n\geq 1}\log(1-x^n)^{-\mu(n)/n}\\ &= \sum_{n\geq 1}\frac{-\mu(n)}{n}\log(1-x^n).\end{aligned}$$

It is the well-known that

$$\log(1+x) = \sum_{k\geq 1}\frac{(-1)^{k-1}}{k}x^k,$$

thus

$$\begin{aligned}\log F(x) &= \sum_{n\geq 1}\frac{-\mu(n)}{n}\sum_{k\geq 1}\left(-\frac{x^{kn}}{k}\right)\\ &= \sum_{n\geq 1}\sum_{k\geq 1}\frac{\mu(n)}{kn}x^{kn}.\end{aligned}$$

The coefficient of $x^m$ in the above is

$$\frac{1}{m}\sum_{d|m}\mu(d),$$

where the sum is over all positive integers $d$ dividing $m$. It is the well-known that

$$\frac{1}{m}\sum_{d|m}\mu(d) = \begin{cases} 1, & m=1\\ 0, & \text{otherwise.}\end{cases}$$

Hence $\log F(x) = x$, therefore $F(x) = e^x$.

**Example 1.2.** From [1]. Find an unique sequence $a_0 = 1$, $a_1, a_2 \ldots$ of real numbers satisfying

$$\sum_{k=0}^{n} a_k a_{n-k} = 1. \tag{1.9}$$

Observe that the left-hand side of the above is a coefficient of convolution of ordinary generating functions. Indeed, let $F(x) = \sum_{n\geq 0} a_n x^n$, then

$$F(x)^2 = \sum_{n\geq 0} x^n = \frac{1}{1-x}.$$

Hence

$$F(x) = (1-x)^{-1/2} = \sum_{n\geq 0}\binom{-1/2}{n}(-x)^n.$$

Therefore the coefficients $a_n$ take a form

$$a_n = (-1)^n \binom{-1/2}{n} = (-1)^n \frac{(-1/2)(-3/2)(-5/2)\cdots(-(2n-1)/2)}{n!}$$
$$= \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!}.$$

**Example 1.3.** From [1]. Verify the identity

$$\sum_{i=0}^{n} \binom{a}{i}\binom{b}{n-i} = \binom{a+b}{n}, \tag{1.10}$$

where $a, b$ and $n$ are nonnegative integers.

Observe that the above might be solved with the help of convolution (1.4) of generating function $F_s(x) = \sum_{i\geq 0} \binom{s}{i} x^i = (1+x)^s$, i.e.,

$$\sum_{k\geq 0}\sum_{i=0}^{n} \binom{a}{i}\binom{b}{n-i} x^k = \left(\sum_{k\geq 0}\binom{a}{k} x^k\right)\left(\sum_{k\geq 0}\binom{b}{k} x^k\right)$$
$$= (1+x)^a(1+x)^b = (1+x)^{a+b}$$
$$= \sum_{k\geq 0}\binom{a+b}{k} x^k.$$

## 2 Binomial posets

**Theorem 1.** *Let $R(P)$ be Reduced Incidence Algebra over binomial poset $P$. Then we have $\phi : R(P) \to \mathbb{C}[[x]]$ given by*

$$\phi(f) = \sum_{n\geq 0} f(n)\frac{x^n}{B(n)}, \tag{2.1}$$

*where $B(n)$ is the total number of maximal chains in $n$-inverval $[x, y]$ of poset $P$.*

**Observation 1.** *Let $f(n)$ be the cardinality of an $n$-interval $[x, y]$ of $P$. Then*

$$\sum_{n\geq 0} f(n)\frac{x^n}{B(n)} = \left(\sum_{n\geq 0}\frac{x^n}{B(n)}\right)^2. \tag{2.2}$$

*Proof.* Notice that $\phi(\zeta) = \sum_{n\geq 0}\frac{x^n}{B(n)}$ and $\zeta^2 = \mathrm{card}[x, y]$. $\square$

**Observation 2.** *If $\mu(n)$ denotes the Möbius function $\mu(x, y)$ for an $n$-interval $[x, y]$ of $P$, then we have*

$$\sum_{n\geq 0} \mu(n)\frac{x^n}{B(n)} = \left(\sum_{n\geq 0}\frac{x^n}{B(n)}\right)^{-1}. \tag{2.3}$$

**Examples:**

1. *An ordinary generating function $F(x) = \sum_{n\geq 0} f(n)x^n$*

$$\sum_{n\geq 0}\binom{t}{n} x^n = (1+x)^t. \tag{2.4}$$

2. *An exponential generating function* $F(x) = \sum_{n\geq 0} f(n)x^n/n!$

$$\sum B(n)\frac{x^n}{n!} = \exp\{e^x - 1\}, \qquad \sum D(n)\frac{x^n}{n!} = \frac{e^{-x}}{1-x}. \tag{2.5}$$

Where $B(n)$ stay here for the Bell numbers and $D(n)$ for the number of permutations of $n$ with no fixed points.

3. *Eulerian generating functions* $\sum_{n\geq 0} x^n/n_q!$, where $n_q! = (1+q)\cdots(1+q+\cdots+q^{n-1})$.

$$\sum_{n\geq 0} f(n)\frac{x^n}{n_q!} = \left(\sum_{n\geq 0}\frac{x^n}{n_q!}\right)^2, \tag{2.6}$$

where $f(n)$ - the total number of subspaces of $V_n(q)$, i.e., $f(n) = \sum_k \binom{n}{k}_q$.

4. *Chromatic generating functions* $F(x) = \sum_{n\geq 0} f(n)\frac{x^n}{q^{\binom{n}{2}}n!}$ for $q \in \mathbb{P}$

$$\sum_{n\geq 0} f(n)\frac{x^n}{2^{\binom{n}{2}}n!} = \left(\sum_{n\geq 0}(-1)^n\frac{x^n}{2^{\binom{n}{2}}n!}\right)^{-1}, \tag{2.7}$$

where $f(n)$ is the number of acyclic digraphs on $n$ vertices.

# References


[1] Richard P. Stanley, *Enumerative Combinatorics Vol.1*, Cambridge University Press 2002.

[2] Herbert S. Wilf, *Generatingfunctionology*, Academic Press, New York 1990.

[3] Andrzej K. Kwaśniewski, *Home Page AKK - http://ii.uwb.edu.pl/akk/.*