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Counting lattice paths

PhD dissertation

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Author's declaration:

aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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the dissertation is ready to be reviewed

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Abstract

Counting lattice paths

Maciej Dziemiańczuk

A lattice path is a finite sequence of points $p_0, p_1, \ldots, p_n$ in $\mathbb{Z} \times \mathbb{Z}$, and a step of the path is the difference between two of its consecutive points, i.e., $p_i - p_{i-1}$. In this thesis, we consider lattice paths running between two fixed points and for which the set of allowable steps contains the vertical step $(0, -1)$ and some number (possibly infinite) of non-vertical steps $(1, k)$, with $k \in \mathbb{Z}$. These paths generalize the well-studied simple directed lattice paths which are composed of only non-vertical steps.

This thesis is divided into two parts. In the first part (Chapter 2), we show that certain families of paths with vertical steps can be coded by weighted simple directed lattice paths (without this vertical step). Several results for paths with vertical steps are obtained and applied to three special families of paths connected with Łukasiewicz, Raney, and Dyck paths. The second part of the thesis (Chapter 3) is devoted to the study of plane multitrees which are defined as weighted unlabeled rooted trees in which the order of sons is significant. We show that there is a one-to-one correspondence between plane multitrees and Raney lattice paths. This correspondence is the main tool to derive several combinatorial and statistical properties of plane multitrees.

Keywords: lattice paths, plane trees, bijective combinatorics.

AMS MS Classification 2000: 05A15, 05A19, 05C30.

Streszczenie

Zliczanie ścieżek kratowych

Maciej Dziemiańczuk

Ścieżka kratowa to skończony ciąg punktów $p_0, p_1, \ldots, p_n$ ze zbioru $\mathbb{Z} \times \mathbb{Z}$, natomiast segment ścieżki to różnica $p_i - p_{i-1}$ dwóch kolejnych punktów ścieżki. W tej rozprawie badamy ścieżki pomiędzy dwoma ustalonymi punktami, dla których zbiór dozwolonych segmentów zawiera segment wertykalny $(0, -1)$ oraz pewną przeliczalną liczbę segmentów niewertykalnych $(1, k)$, gdzie $k \in \mathbb{Z}$. Ścieżki te uogólniają dobrze znane z literatury tak zwane proste ścieżki skierowane (ang. *simple directed lattice paths*), które składają się jedynie z segmentów niewertykalnych.

Niniejsza rozprawa podzielona jest na dwie części. W pierwszej części (Rozdział 2), pokazujemy, że pewne rodziny ścieżek z segmentami wertykalnymi możemy kodować za pomocą ważonych prostych ścieżek skierowanych. Zaprezentowany zostanie szereg rezultatów dla ogólnego przypadku, które zostaną następnie zastosowane dla trzech szczególnych rodzin ścieżek związanych ze ścieżkami Łukasiewicza, Raneya i Dycka. Druga część rozprawy (Rozdział 3) poświęcona jest badaniu pewnych własności multidrzew porządkowych, które definiuje się jako nieetykietowane ukorzenione drzewa, w których dodatkowo ustala się porządek synów oraz krawędziom przypisuje liczby naturalne. Zamiast zajmować się bezpośrednio tymi strukturami, pokażemy bijekcję pomiędzy drzewami porządkowymi a ścieżkami Raneya. Dzięki tej bijekcji otrzymany zostanie szereg kolejnych wynikow dla multidrzew.

Słowa kluczowe: ścieżki kratowe, drzewa porządkowe, kombinatoryka bijektywna.

Klasyfikacja AMS MSC 2000: 05A15, 05A19, 05C30.

Contents

Notation

$\mathbb{Z}$	The set of integers.
$\mathbb{N}$	The set of nonegative integers.
$[n]$	The set $\{1, 2, \ldots, n\}$ for $n \geq 1$, and $[0] = \emptyset$.
$n^{\underline{m}}$	$= n(n-1)\cdots(n-m+1)$ for $m \in \mathbb{Z}$, and $n^{\underline{0}} = 1$.
$[x^n]f(x)$	The coefficient of x^n in the power series expansion of $f(x)$.
C_n	The nth Catalan number (p. 2).
V	The vertical step $(0, -1)$ (p. 5).
S_k	The non-vertical step $(1, k)$ for $k \in \mathbb{Z}$ (p. 5).
U_k	The up step $(1, k)$ for $k \geq 0$ (p. 5).
D_k	The down step $(1, -k)$ for $k \geq 1$ (p. 5).
Ω	The set of all non-vertical steps $\{S_k : k \in \mathbb{Z}\}$ (p. 5).
Σ	A set of steps.
$\Sigma_{\geq k}$	$= \Sigma \cap \{S_k, S_{k+1}, \ldots\}$ for any Σ and $k \in \mathbb{Z}$ (p. 5).
Λ	A set of steps satisfying $\Lambda \subseteq \{V, S_N, S_{N-1}, \ldots\}$ and $V, U_N \in \Lambda$.
Γ	$= (\Lambda \setminus \{V\}) \cup \{U_N, U_{N-1}, \ldots, U_0, D_1\}$ (p. 17).
λ	The lattice path with zero steps (the empty path).
μ, π, γ	Lattice paths.
f, g, ρ, κ	Functions.
$\#Steps(S \in \mathcal{P})$	The number of occurrences of the step S in paths of $\mathcal{P}$ (p. 6).
$\#Steps(\mathcal{P})$	The number of all steps in paths of $\mathcal{P}$ (p. 6).
Π_μ, $\Pi_\mu(i)$	List of integer lattice points (p. 14 and p. 51).
$\mathrm{il}_\pi(i)$, $\mathrm{el}_\pi(i)$	The initial and the ending levels of the ith step of π (p. 51).
$\epsilon(\pi)$	The function over up steps of the lattice path π (p. 63).

$\mathcal{P}_\Sigma(n,-m)$	The family of m-primary Σ-paths running from $(0,0)$ to $(n,-m)$ (p. 6).
$P_{\Lambda,m}(x)$	The generating function of the numbers $\lvert\mathcal{P}_\Lambda(n,-m)\rvert$ (p. 37).
$\mathcal{F}_\Sigma(n,m)$	The family of free Σ-paths running from $(0,0)$ to (n,m) (p. 6).
$F_\Lambda(x,y)$	The generating function of the numbers $\lvert\mathcal{F}_\Lambda(n,m)\rvert$ (p. 34).
$\mathcal{W}_\Gamma^\Lambda(n,-m)$	The family of weighted m-primary Γ-paths (p. 18).
$\mathcal{R}(n)$	The family of Raney paths of length n (p. 7).
$\mathcal{R}_N(n)$	The family of N-Raney paths of length n (p. 7).
$R(m,n,d)$	The number of (m,n,d)-Raney sequences (p. 63).
$\mathcal{T}(n)$	The family of plane multitrees with n nodes (p. 7).
$\mathcal{T}_N(n)$	The family of N-ary plane multitrees with n nodes (p. 8).
$T_N(n)$	The size of $\mathcal{T}_N(n)$ (p. 8).
$L_N(n,k)$	The number of trees in $\mathcal{T}_N(n)$ which have exactly k leaves (p. 66).
$E_N(n,k)$	The number of trees in $\mathcal{T}_N(n)$ which have exactly k edges (p. 68).
$G_N(n,d)$	The number of trees in $\mathcal{T}_N(n)$ whose root has outdegree d (p. 69).
$M_N(n,d)$	The number of nodes with outdegree d in all trees in $\mathcal{T}_N(n)$ (p. 69).
$B(N,n,d)$	$= M_N(n,d)/(nT_N(n)$ (p. 71).
$J(N,n)$	$= \sum_{d=0}^{N} dM_N(n,d)/(nT_N(n))$ (p. 72).
$\mathcal{C}(m,d,j)$	A set of certain compositions of the number j (p. 17).
$\mathcal{H}_\Lambda(m,d,k)$	A set of certain pairs (p. 17).
$w_\mu(i)$	The maximal weight of the ith step in a lattice path μ (p. 18).
$\mathrm{sons}_T(v)$	The list of sons of a node v in a plane multitree T (p. 50).
$\alpha(\psi)$	The number of additional edges in ψ (p. 61).

Chapter 1

Introduction

A *lattice path* (or simply a path) is a finite sequence of points $p_0, p_1, \ldots, p_n$ in $\mathbb{Z} \times \mathbb{Z}$. A *step* of the path is the difference between two of its consecutive points, i.e., $p_i - p_{i-1}$, for $1 \le i \le n$. The lattice path can also be represented by the initial point p_0 and the sequence of its steps $s_1, s_2, \ldots, s_n$, which uniquely determine the remaining points of the path. For instance, the path from Figure 1.1 is

$$\big((0,0),(1,3),(2,1),(3,1),(4,0),(5,-1),(6,1),(7,1),(8,0)\big),$$

whose step representation is the initial point $(0,0)$ and the following sequence of steps:

$$\big((1,3),(1,-2),(1,0),(1,-1),(1,-1),(1,2),(1,0),(1,-1)\big).$$

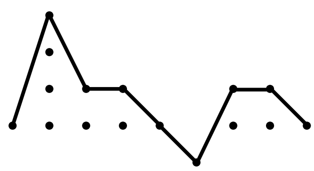

FIGURE 1.1: A lattice path running from $(0,0)$ to $(8,0)$.

The literature on lattice paths is very rich. Humphreys [23] refers to more than two hundred crucial articles. Most of them are related to path enumeration problems and relationships with other structures.

Some of the most well-known families of lattice paths are those that consist of two types of steps: $(1,1)$ and $(1,-1)$. These paths are called *Dyck paths.* In 1878, Whitworth [40] used them to describe various combinatorial problems. In 1887, Bertrand [5] formulated the famous ballot problem, which can be translated into a question about the number of Dyck paths running from $(0,0)$ to $(u+d, u-d)$, where $u > d$, and that do not touch the x-axis except at the initial point. An example of such a path, for $u = 6$ and $d = 4$, is given in Figure 1.2. André [1] solved this problem and showed that the number of such paths is equal to

$$\binom{u+d}{u} - 2\binom{u+d-1}{u} = \binom{u+d-1}{u}\frac{u-d}{d}. \tag{1.1}$$

Setting $u = n+1$ and $d = n$ in (1.1), we obtain the number of Dyck paths running from $(0,0)$ to $(2n,0)$ and that never go below the x-axis. Their number is equal to the Catalan number C_n given by

$$C_n = \frac{1}{n+1}\binom{2n}{n} \qquad (n \geq 0).$$

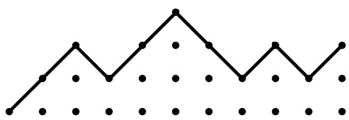

FIGURE 1.2: A Dyck path running from $(0,0)$ to $(10,2)$.

The sequence of consecutive Catalan numbers is denoted by A000108 in OEIS [32], and it starts with the following numbers:

$$(C_n)_{n\geq 0} = (1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, \ldots).$$

Simple generalizations of Dyck paths are *Motzkin paths*, which consist of three types of steps: $(1,1)$, $(1,0)$, and $(1,-1)$. The number of Motzkin paths running from $(0,0)$ to $(n,0)$ that do not go below the x-axis is called the nth *Motzkin number* [12]. Motzkin considered these numbers in terms of counting chords. Namely, the nth Motzkin number is the number of ways of drawing at most $n/2$ non-intersecting chords between n fixed points on a circle (see Figure 1.3). For $n \geq 1$, this number is equal to

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k.$$

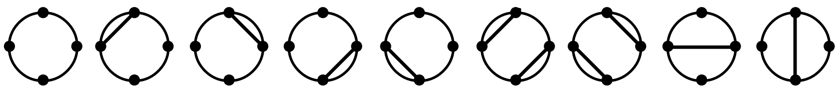

FIGURE 1.3: All the ways of drawing at most two non-intersecting chords between four fixed points on a circle.

Donaghey and Shapiro [12] provided a representative selection of 14 situations wherein Motzkin numbers occur in connection with other combinatorial structures such as Dyck paths, sequences of parentheses, trees with loops, and bipartite graphs. For example, they showed that the number of legal sequences of parentheses that contain exactly $2n$

symbols (and) in total and that do not contain a subsequence $((\sigma))$, where σ is a legal sequence of parentheses, is equal to the $(n-1)$th Motzkin number. For example, there are exactly 9 such legal sequences of 10 parentheses, i.e.,

$$()()()()(), \quad (()())()(), \quad ()(()())(), \quad ()()(()()), \quad (()()())(),$$
$$()(()()()), \quad (()()()()), \quad ((()())()), \quad (()(()())).$$

Łukasiewicz paths, named after the Polish logician Jan Łukasiewicz (1878-1956), represent another generalization of Dyck paths. A *Łukasiewicz path* is a lattice path in which the set of allowable steps contains all steps of the form $(1, k)$ for $k \geq -1$. This structure was studied by, among others, Viennot [38] and Stanley [33]. Using decompositions of these paths and generating functions (see Flajolet and Sedgewick [19]), one can show that the number of Łukasiewicz paths running from $(0, 0)$ to $(n, 0)$ that do not go below the x-axis is equal to the Catalan number C_n. These paths are useful in the analysis of algorithms because of their close relationship with plane trees, which will be discussed in Chapter 3 of this thesis.

The structure that generalizes all the above-mentioned paths is lattice paths that consist of steps $(1, k)$ for any $k \in \mathbb{Z}$. They are called *simple directed paths* (see, e.g., Banderier and Flajolet [2]). They are used as models in the analysis of algorithms and dynamic data structures [27]. They are also used to describe the behavior of stack structures. A unified approach to simple directed paths was developed by Banderier and Flajolet [2]. They showed that the counting generating functions of such paths are algebraic functions. They also described the asymptotic behavior of these numbers using the method of singularity analysis.

Weighted (or colored) paths are used in several applications. Deutsch and Shapiro [11] studied weighted Motzkin paths. Based on their paper, one can deduce that there is a bijection between a family of certain weighted Motzkin paths and non-weighted Dyck paths. They also established connections between weighted Motzkin paths, Schröder paths, and other graph counting problems. Chen et al. [7, 39] studied weighted Motzkin paths in the context of Riordan arrays and partitions of sets. They proved that there is a bijection between a certain subfamily of weighted Motzkin paths running from $(0, 0)$ to $(n, 0)$ and the set of noncrossing linked partitions of the set $\{1, 2, \ldots, n+1\}$ for $n \geq 0$. They also showed that there is a bijection between weighted Motzkin paths and non-weighted Schröder paths.

A generalization of weighted Łukasiewicz paths was studied by Varvak [37]. She showed that there are several one-to-one correspondences between appropriate weighted Łukasiewicz paths and multipermutations, partitions of sets, idempotent functions, and trees. The main tool that she used was a connection between weighted Łukasiewicz paths and continued fractions (see also Flajolet [17]). Hennessy [22, Sec. 5.3] showed that there are bijections between certain families of Łukasiewicz, Schröder, and Motzkin paths.

Lattice paths are also used in the theory of formal languages [13, 18, 20, 28, 29]. Lattice paths also appear in the theory of queues [6] and in methods of randomly generating structured objects [4]. Finally, they are used in physics [36] and probability theory [35].

In Chapter 3, we use lattice paths to study the combinatorial and statistical properties of plane multitrees. Plane trees are well-known structures in combinatorics [8, 9, 11, 25, 30]. There are several equivalent definitions of plane trees, but they are mostly defined as unlabeled rooted trees in which the order of sons is significant. There is a bijection between the family of all plane trees with n nodes and the set of Łukasiewicz paths running from $(0, 0)$ to $(n, -1)$ in which only the ending point lies below the x-axis (see Flajolet and Sedgewick [19, Sec. I.5.1]). This bijection implies that the number of plane trees with n nodes is equal to the Catalan number C_{n-1}. It is also well known that the number of plane trees with n nodes and k leaves is equal to the Narayana number (see Dershowitz and Zaks [8]) given by

$$\frac{1}{n-1}\binom{n-1}{k}\binom{n-1}{k-1}.$$

Dershowitz and Zaks [8] also showed that the expected number of leaves in a plane tree with n nodes is $n/2$ and that the expected number of outcoming edges from the root in this tree is $3(n-1)/(n+1)$.

As we have already noted, plane trees are used in computer science due to their close relationship with Łukasiewicz codes and polish prefix notation (see, e.g., Sedgewick i Flajolet [19, Sec. I.5.3]). Computer compilers use such trees as structures to parse expressions (see Knuth [26, Sec. 2.3]). For instance, the graphical representation of the expression $(c_1)((c_2)(c_3))((c_4)(c_5))$ is the plane tree given on the left-hand side of Figure 1.4. Let us now consider the case when $c_2 = c_4$ and $c_3 = c_5$. The expression can be rewritten as $(c_1)((c_2)(c_3))^2$. To represent these exponents, we assign weights to the edges of the plane tree. These weights are positive integers and can be drawn as multiple edges; see the right-hand side of Figure 1.4.

The first mention of plane multitrees is credited to R. Bacher (see the description of the sequence A002212 in OEIS [32]). He showed that the number of plane multitrees with n edges is equal to

$$\sum_{k=0}^{n} C_k \binom{n-1}{k-1}, \qquad (n \geq 0),$$

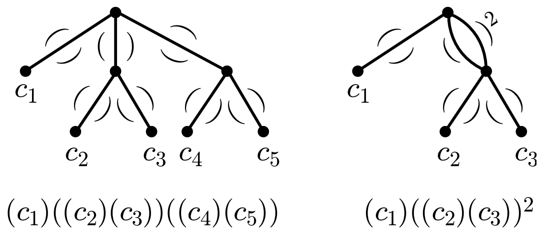

FIGURE 1.4: A plane tree and multitree with corresponding sequences of parentheses.

Let us present a few first members of the above sequence, from $n = 0$ to $n = 10$,

$$(1, 1, 3, 10, 36, 137, 543, 2219, 9285, 39587, 171369).$$

This sequence is denoted by A002212 in OEIS [32].

1.1 Definitions

Before we present the results of this dissertation, we introduce our notation. Throughout the thesis, we will consider lattice paths that consist of steps:

- $V = (0, -1)$ (*vertical step*),
- $S_k = (1, k)$ for $k \in \mathbb{Z}$ (*non-vertical step*).

For convenience, we denote

- $U_k = (1, k)$ for $k \geq 0$ (*up step*),
- $D_k = (1, -k)$ for $k \geq 1$ (*down step*),
- $\Omega = \{S_k : k \in \mathbb{Z}\}$,
- for any set of steps Σ and $k \in \mathbb{Z}$, we write $\Sigma_{\geq k} = \Sigma \cap \{S_k, S_{k+1}, \ldots\}$.

Definition 1.1. Let Σ be a fixed subset of $\Omega \cup \{V\}$. A Σ-*path* π is a finite sequence of points $p_0, p_1, \ldots, p_n$ in $\mathbb{Z} \times \mathbb{Z}$ such that $p_i - p_{i-1} \in \Sigma$ for $i \in \{1, \ldots, n\}$. For simplicity, we will represent the path π by the word $\pi_1 \pi_2 \cdots \pi_n$ over the alphabet Σ, where $\pi_i = p_i - p_{i-1}$. The starting point of π is assumed to be $p_0 = (0, 0)$ or is given by the context. We assume that there is one lattice path, denoted by λ, with no steps; this path will be called the *empty path*. We identify an element $S \in \Sigma$ with the Σ-path consisting of one step S. For $k \geq 1$, we will write S^k to denote k consecutive steps $S \in \Sigma$ and $S^0 = \lambda$. The *length* of a path is the number of steps that it contains. For $l \in \mathbb{Z}$, by *level* l we mean the line $y = l$.

Example. Let $\Sigma = \{U_4, U_2, U_1, U_0, D_1, V\}$. An example of a Σ-path running from $(0, 0)$ to $(11, -1)$ is $U_4 V U_1^2 D_1 V U_0 V^2 U_2 V^3 U_2 D_1 U_0 U_1 V^2 D_1$ whose graphical representation is given in Figure 1.5.

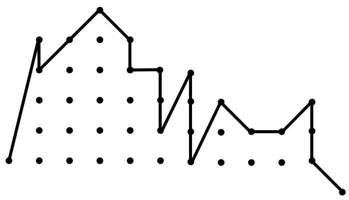

FIGURE 1.5: A 1-primary path running from $(0,0)$ to $(11,-1)$.

Definition 1.2. An *m-primary Σ-path* is a Σ-path running from $(0,0)$ to some $(n,-m)$, with $n \geq 0$ and $m \geq 0$, whose all points, except the possibly last one, lie on or above the horizontal axis. We call a path *primary* if it is an m-primary with $m \geq 1$. We will denote by $\mathcal{P}_\Sigma(n,-m)$ the family of all m-primary Σ-paths running from $(0,0)$ to $(n,-m)$. Additionally, we assume that $\mathcal{P}_\Sigma(0,0) = \{\lambda\}$, where λ is the empty path with zero steps, and $\mathcal{P}_\Sigma(0,-m)$ is the empty set for $m \geq 1$.

Example. Let $\Sigma = \{U_1, U_0, D_1, D_2\}$. All 1-primary Σ-paths in $\mathcal{P}_\Sigma(3,-1)$ are given in Figure 1.6.

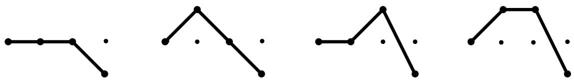

FIGURE 1.6: All 1-primary paths running from $(0,0)$ to $(3,-1)$ whose steps belong to the set $\{U_1, U_0, D_1, D_2\}$.

Definition 1.3. A *free* Σ-path is a Σ-path running from $(0,0)$ to some $(n,-m)$ with $n \geq 1$, $m \in \mathbb{Z}$. We will denote by $\mathcal{F}_\Sigma(n,-m)$ the family of all free Σ-paths running from $(0,0)$ to $(n,-m)$.

Example. All free Σ-paths of $\mathcal{F}_\Sigma(3,-1)$, where $\Sigma = \{U_1, U_0, D_1, D_2\}$, are given in Figure 1.7.

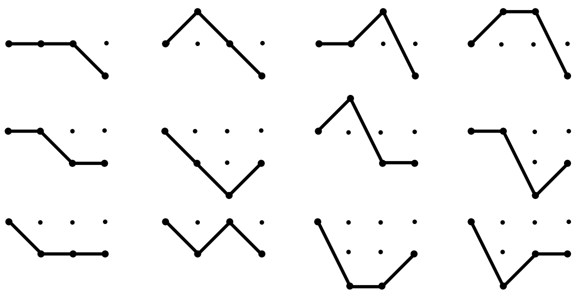

FIGURE 1.7: All free paths running from $(0,0)$ to $(3,-1)$ whose steps belong to the set $\{U_1, U_0, D_1, D_2\}$.

Definition 1.4. Let S be a step in Σ. We denote by $\#Steps(S \in \mathcal{P}_\Sigma(n,-1))$ the total number of occurrences of the step S in all paths of $\mathcal{P}_\Sigma(n,-1)$ and by $\#Steps(\mathcal{P}_\Sigma(n,-1))$ the total number of steps in all paths of $\mathcal{P}_\Sigma(n,-1)$.

Definition 1.5. A *Raney path* of length n is an Ω-path running from $(0,1)$ to $(n,0)$, $n \geq 1$, in which only the ending point of the path lies below level 1. For $N \geq 0$, an N*-Raney path* of length n is a Raney path that does not contain steps U_k, with $k > N$. Let $\mathcal{R}(n)$ (respectively $\mathcal{R}_N(n)$) denote the family of all Raney (respectively N-Raney) paths of length n.

Remark. It is worth noting that there is a trivial bijection between $\mathcal{R}(n)$ and $\mathcal{P}_\Omega(n,-1)$. The Raney paths start at $(0,1)$ instead of $(0,0)$ for reasons that will become clear in Chapter 3.

Example. All 1-Raney paths of length 3 are given in Figure 1.8.

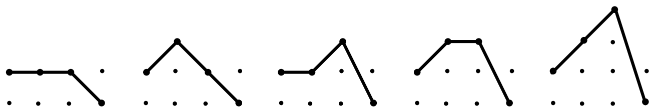

FIGURE 1.8: All 1-Raney paths of length 3.

Definition 1.6. A *plane tree* T is a pair (V,E), where $V = \{1,2,\ldots,n\}$ is the set of vertices for some n called the *size* of T, and $E \subset V \times V$ is a set of arcs satisfying the following conditions:

(i) if $(u,v) \in E$, then $u < v$,

(ii) for every vertex $v \neq 1$, there is exactly one vertex u such that $(u,v) \in E$,

(iii) if $\{(u,v_1),(u,v_2)\} \subseteq E$, $v_1 < v_2$, and w is a descendant of v_1, then $w < v_2$.

For every $v \in V$, we denote by T_v the subtree of T rooted in v.

Remark. There are several equivalent definitions of plane trees in the literature. Kemp [24] defined a plane tree as a rooted unlabeled tree that has been embedded in the plane such that the relative order of the subtrees at each branch is part of its structure. Flajolet and Sedgewick [19] defined a plane tree recursively as a root to which a (possibly empty) sequence of trees is attached. These objects also appear in the literature as ordered trees (see, e.g., Deutsch and Shapiro [11], Dershowitz and Zaks [8]), where the term ordered refers to the order of sons. We shall show in Section 3.1 that our definition agrees with these definitions.

Example. All plane trees with four nodes are given in Figure 1.9.

Definition 1.7. A *plane multitree* $T = (V,E,w)$ is a plane tree (V,E) in which every arc e in E is labeled by a positive integer $w(e)$ called the *weight* of the arc. Let us denote by $\mathcal{T}(n)$ the family of all plane multitrees with n nodes.

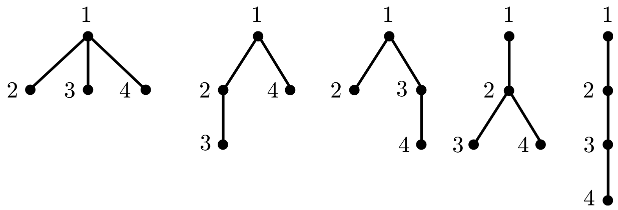

FIGURE 1.9: All plane trees with four nodes. The root of each tree is on the top, and the arcs are directed downward.

Throughout this dissertation, we will use an equivalent representation of a plane multitree $T = (V, E, w)$ as a rooted tree, in which

(i) the root is the node with label one,

(ii) every internal node has assigned the order of its sons from left to right based on the natural order of labels of its sons,

(iii) the set of weighted arcs is represented by the multiset (E, w) (set with repetitions) of arcs in the following manner. Every arc $e = (u, v)$ weighted by a positive integer $w(e)$ is transformed into $w(e)$ nonweighted arcs (u, v) called *edges*. Moreover, one (the leftmost) of these edges will be called the *main edge*, and the remaining $w(e) - 1$ edges will be called *additional edges*.

Example. A plane tree $T = (V, E, w)$ with weighted arcs (left) and its corresponding plane multitree with nonweighted edges (right) are given in Figure 1.10. In this case, we have $V = \{1, 2, \ldots, 9\}$, and the multiset (E, w) is

$$\{(1,2),(1,2),(1,6),(1,7),(2,3),(2,3),(2,3),(3,4),(3,5),(7,8),(7,9),(7,9)\}.$$

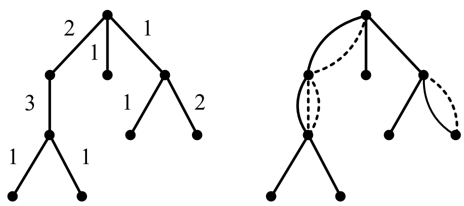

FIGURE 1.10: A plane tree with weighted arcs (left) and its corresponding plane multitree with main edges drawn using solid lines and additional edges drawn using dashed lines (right).

Definition 1.8. Suppose that $v \in V$. Let $\operatorname{odeg}(v)$ denote the total number of outgoing edges from v to its children. We call $\operatorname{odeg}(v)$ the *outdegree* of v. In other words, $\operatorname{odeg}(v)$ is the sum of weights of all arcs outgoing from v. A plane multitree, in which $\operatorname{odeg}(v) \le N$ for all $v \in V$, will be called the *N-ary plane multitree*. For $n \ge 0$, let $\mathcal{T}_N(n)$ denote the family of N-ary plane multitrees with n nodes, and let $T_N(n)$ denote the size of this family.

Example. We have $T_2(4) = 14$, and all trees of $\mathcal{T}_2(4)$ are given in Figure 1.11.

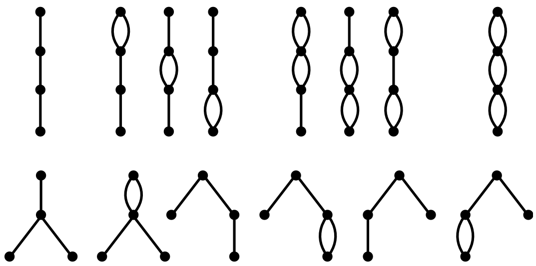

FIGURE 1.11: All 2-ary plane multitrees with four nodes.

1.2 Results of the thesis

Results of Chapter 2

Chapter 2 is devoted to the study of two families of primary and free Λ-paths, where Λ is an arbitrary set of lattice steps satisfying $\Lambda \subseteq \{V, S_N, S_{N-1}, \ldots\}$ and $S_N, V \in \Lambda$ for any fixed $N \geq 0$. The results of this part originate from the paper [16].

In Section 2.2, we define a special family $\mathcal{W}_\Gamma^\Lambda(n, -m)$ of weighted m-primary Γ-paths, where

$$\Gamma = (\Lambda \setminus \{V\}) \cup \{U_N, \ldots, U_0, D_1\}.$$

In Section 2.3, we show (Theorem 2.10) that there is a bijection between the family $\mathcal{P}_\Lambda(n, -m)$ of m-primary Λ-paths and $\mathcal{W}_\Gamma^\Lambda(n, -m)$ for every $m, n \geq 0$.

This implies that the additional vertical steps V in the paths of $\mathcal{P}_\Lambda(n, -m)$ can be coded using the weights of steps in paths without V. From the combinatorial point of view, lattice paths in $\mathcal{W}_\Gamma^\Lambda(n, -m)$ have a simpler structure than do the paths in $\mathcal{P}_\Lambda(n, -m)$. Recall that the paths of $\mathcal{W}_\Gamma^\Lambda(n, -m)$ are simple directed paths, and note that they are essentially one-dimensional objects. There are many results for simple directed paths that have already been described in the literature. The classical work here is the paper of Banderier and Flajolet [2]. Therefore, using the bijection mentioned above, we can apply some of these results to Λ-paths. For instance, simple directed paths can be easily decomposed into shorter subpaths, and this decomposition property provides a straightforward method of calculating generating functions that count these paths.

In Section 2.4, we establish the following connections between 1-primary Λ-paths and free Λ-paths. We prove (Theorem 2.17 and Theorem 2.18) that for $n \geq 1$, we have

$$\begin{aligned} |\mathcal{P}_\Lambda(n, -1)| &= \frac{1}{n}\Big(\big|\mathcal{F}_\Lambda(n, -1)\big| - \big|\mathcal{F}_\Lambda(n, 0)\big|\Big) \\ &= \frac{1}{n}\sum_{j=0}^{Nn+1}\binom{n+j-1}{j}\big|\mathcal{F}_{\Lambda\setminus\{V\}}(n, j-1)\big|. \end{aligned}$$

Further, we show (Theorem 2.19 and Theorem 2.20) that for $n \geq 1$,

$$\begin{aligned}
\#Steps(V \in \mathcal{P}_\Lambda(n,-1)) &= |\mathcal{F}_\Lambda(n,0)|, \\
\#Steps(S_k \in \mathcal{P}_\Lambda(n,-1)) &= |\mathcal{F}_\Lambda(n-1,-k-1)|, \qquad (S_k \in \Lambda), \\
\#Steps(\mathcal{P}_\Lambda(n,-1)) &= |\mathcal{F}_\Lambda(n,-1)|.
\end{aligned}$$

In Section 2.5, we provide various results for Λ-paths. We show (Theorem 2.22) that the numbers of free and 1-primary Λ-paths are given by the following formulas:

$$\begin{aligned}
|\mathcal{F}_\Lambda(n,m)| &= [x^{Nn-m}]\frac{1}{(1-x)^{n+1}}\Big(\sum_{S_k \in \Lambda} x^{N-k}\Big)^n, \qquad (n \geq 1, m \in \mathbb{Z}), \\
|\mathcal{P}_\Lambda(n,-1)| &= \frac{1}{n}[x^{Nn+1}]\frac{1}{(1-x)^{n}}\Big(\sum_{S_k \in \Lambda} x^{N-k}\Big)^n, \qquad (n \geq 1).
\end{aligned}$$

We also derive certain statistical properties of Λ-paths. Any Γ-path running from $(0,0)$ to (n,m) has exactly n steps. The number of steps in a Λ-path running between the same points is equal to or greater than n. We show (Corollary 2.23) that the expected number of steps in a path of $\mathcal{P}_\Lambda(n,-1)$ is equal to

$$n \cdot \frac{|\mathcal{F}_\Lambda(n,-1)|}{|\mathcal{F}_\Lambda n,-1)| - |\mathcal{F}_\Lambda(n,0)|}, \qquad (n \geq 1).$$

In Section 2.6, we use the bijection from Section 2.3 to derive a functional equation for the generating function $P_{\Lambda,m}(x) = \sum_{n\geq 0} |\mathcal{P}_\Lambda(n,-m)|x^n$. We show (Theorem 2.27) that

$$\begin{aligned}
P_{\Lambda,0}(x) &= 1 + \delta_{\Lambda,0} x P_{\Lambda,0}(x) + x P_{\Lambda,0}(x) \sum_{k=1}^{N}\sum_{d=1}^{k} |\mathcal{H}_\Lambda(0,d,k)| \sum_{M} \prod_{j=1}^{d} P_{\Lambda,m_j}(x), \\
P_{\Lambda,m}(x) &= \delta_{\Lambda,m} x + x \sum_{k=0}^{N}\sum_{d=1}^{k+1} |\mathcal{H}_\Lambda(m,d,k)| \sum_{M} \prod_{j=1}^{d} P_{\Lambda,m_j}(x), \qquad (m \geq 1),
\end{aligned}$$

for some constants $\delta_{\Lambda,m}$ and $|\mathfrak{P}_\Lambda(m,d,k)|$ depending on Λ (see Section 2.6).

In Sections 2.7 – 2.9, we consider three cases for the set of steps Λ. These examples are connected with the well-known families of lattice paths from the literature. We apply to them the general results from the previous sections and see that several of the examples take on a simple form. Namely, for fixed $N, K \geq 0$, we consider

1. Łukasiewicz paths with the set of steps $\Gamma_1 = \{U_N, U_{N-1}, \ldots, U_0, D_1\}$ and generalized Łukasiewicz paths with vertical steps $\Lambda_1 = \{V, U_N, U_{N-1}, \ldots, U_0, D_1\}$ (Section 2.7),

2. Raney paths with the set of steps $\Gamma_2 = \{U_N, U_{N-1}, \dots, U_0, D_1, D_2, \dots\}$ and generalized Raney paths with vertical steps $\Lambda_2 = \{V, U_N, U_{N-1}, \dots, U_0, D_1, D_2, \dots\}$ (Section 2.8),
3. Paths with the set of steps $\Gamma_3 = \{U_N, \dots, U_0, D_1, D_K\}$ and generalized Dyck paths with vertical steps $\Lambda_3 = \{V, U_N, D_K\}$ (Section 2.9),

In Section 2.7, we show (Theorem 2.30) that the functional equation for the generating function $P_{\Lambda_1,m}(x)$ that counts paths in $\mathcal{P}_{\Lambda_1}(n,-m)$ according to n is

$$\begin{aligned} P_{\Lambda_1,0}(x) &= 1 + xP_{\Lambda_1,0}(x)\sum_{k=0}^{N}(1+P_{\Lambda_1,1}(x))^k, \\ P_{\Lambda_1,1}(x) &= x\sum_{k=0}^{N+1}(1+P_{\Lambda_1,1}(x))^k. \end{aligned}$$

We show (Theorem 2.34) that for $m \in \mathbb{Z}$ and $n \geq 1$,

$$\begin{aligned} |\mathcal{F}_{\Lambda_1}(n,m)| &= \sum_{k=0}^{\lfloor \frac{Nn-m}{N+2} \rfloor} (-1)^k \binom{n}{k}\binom{(N+2)(n-k)-m}{2n}, \\ |\mathcal{P}_{\Lambda_1}(n,0)| &= (-1)^n + \sum_{j=1}^{n}\sum_{k=0}^{\lfloor \frac{Nj+1}{N+2} \rfloor} \frac{(-1)^{k+n-j}}{j}\binom{j}{k}\binom{(N+2)(j-k)}{2j-1}, \\ |\mathcal{P}_{\Lambda_1}(n,-1)| &= \frac{1}{n}\sum_{k=0}^{\lfloor \frac{Nn+1}{N+2} \rfloor} (-1)^k \binom{n}{k}\binom{(N+2)(n-k)}{2n-1}. \end{aligned}$$

In Section 2.8, we show (Theorem 2.37) that

$$\begin{aligned} |\mathcal{F}_{\Lambda_2}(n,m)| &= \binom{(N+2)n-m}{2n}, \qquad (n \geq 0, m \in \mathbb{Z}), \\ |\mathcal{P}_{\Lambda_2}(n,-1)| &= \frac{1}{n}\binom{(N+2)n}{2n-1}, \qquad (n \geq 1). \end{aligned}$$

We show that the expected number of vertical steps in a path in $\mathcal{P}_{\Lambda_2}(n,-1)$ is equal to $(Nn+1)/2$ and that the expected number of all steps in a path in $\mathcal{P}_{\Lambda_2}(n,-1)$ is equal to $((N+2)n+1)/2$.

In Section 2.9, we show (Theorem 2.40) that for all $m \in \mathbb{Z}$ and $n \geq 1$, we have

$$\begin{aligned} |\mathcal{F}_{\Lambda_3}(n,-m)| &= \sum_{k=0}^{\lfloor \frac{Nn+m}{N+K} \rfloor} \binom{n}{k}\binom{n(N+1)-k(N+K)+m}{n}, \\ |\mathcal{P}_{\Lambda_3}(n,-1)| &= \frac{1}{n}\sum_{k=0}^{\lfloor \frac{Nn+1}{N+K} \rfloor} \binom{n}{k}\binom{n(N+1)-k(N+K)}{n-1}. \end{aligned}$$

Results of Chapter 3

Chapter 3 is devoted to the study of the combinatorial and statistical properties of plane multitrees. The results of this part originate from the paper [15]. The main tool that will be used in Chapter 3 is a bijection between plane multitrees and Raney paths. Namely, in Section 3.2, we show (Theorem 3.8) that for all $N \geq 0$ and $n \geq 1$, there is a bijection between the family $\mathcal{R}_N(n)$ of N-Raney paths of length n and the family $\mathcal{T}_{N+1}(n)$ of $(N+1)$-ary plane multitrees with n nodes. From the combinatorial view of point, Raney paths have a simpler structure than do plane multitrees. Therefore, the obtained results for Raney paths will be translated to the corresponding properties of plane multitrees.

In Section 3.3, we define a bijection between N-Raney paths of length n and the $(N-1, n, 1)$-Raney sequences (Lemma 3.23). Using this bijection, we show (Theorem 3.24) that the number $T_N(n)$ of N-ary plane multitrees with n nodes is equal to

$$T_N(n) = \frac{1}{n}\binom{Nn}{n-1}, \qquad (N \geq 1, n \geq 1).$$

In Sections 3.4 – 3.7, we use the two above-mentioned bijections to obtain certain results for plane multitrees. Namely, for the family $\mathcal{T}_N(n)$ of all N-ary plane multitrees with n nodes, we consider the following numbers:

1. $L_N(n,k)$ = number of such trees with exactly k leaves,
2. $E_N(n,k)$ = number of such trees with exactly k edges,
3. $G_N(n,d)$ = number of such trees in which the root has d outgoing edges, and
4. $M_N(n,d)$ = total number of nodes that have d outgoing edges in all such trees.

Let $0^d = 0$ for $d > 0$, and $0^0 = 1$, we show that

$$L_N(n,k) = \frac{1}{n}\binom{n}{k}\sum_{s=0}^{n-k}(-1)^s\binom{n-k}{s}\binom{N(n-k-s)}{n-1}, \qquad \text{(Th. 3.26)}$$

$$E_N(n,k) = \frac{1}{n}\sum_{i=1}^{n-1}\sum_{j=0}^{i}(-1)^j\binom{n}{i}\binom{i}{j}\binom{k-i}{n-i-1}\binom{k-jN-1}{i-1}, \qquad \text{(Th. 3.28)}$$

$$G_N(n,d) = \frac{N-1+d}{(N-1)(n-1)+d}\binom{N(n-1)+d-2}{n-2}, \qquad \text{(Th. 3.30)}$$

$$M_N(n,d) = \binom{N(n-1)+d-1+0^d}{n-2+0^d}, \qquad \text{(Th. 3.31)}$$

In Section 3.7, we study the statistical properties of plane multitrees. We prove (Theorem 3.34) that

$$\lim_{N\to\infty}\lim_{n\to\infty}\frac{M_N(n,0)}{nT_N(n)} = \lim_{n\to\infty}\lim_{N\to\infty}\frac{M_N(n,0)}{nT_N(n)} = \frac{1}{e}.$$

Chapter 2

Lattice paths with vertical steps

This chapter is devoted to the study of lattice paths consisting of non-vertical steps $S_k = (1, k)$ for $k \in \mathbb{Z}$ and vertical step $V = (0, -1)$. We fix a nonnegative integer N and take an arbitrary set of steps Λ satisfying $\Lambda \subseteq \{V, S_N, S_{N-1}, \ldots\}$ and $S_N, V \in \Lambda$. In Sections 2.1 – 2.2, we give some preliminary properties of primary paths in $\mathcal{P}_\Lambda(n, -m)$ and define the family $\mathcal{W}_\Gamma^\Lambda(n, -m)$ of weighted m-primary Γ-paths. In Section 2.3, we show that there is a bijection between $\mathcal{P}_\Lambda(n, -m)$ and $\mathcal{W}_\Gamma^\Lambda(n, -m)$ for every $m, n \geq 0$. In Section 2.4, we establish some connections between primary and free Λ-paths. In Sections 2.5 – 2.6, we provide some results for Λ-paths. In Sections 2.7 – 2.9, we apply the general results from the previous sections to three special families of lattice paths that contain vertical steps.

2.1 Decomposition of primary paths

Let Σ be a set of steps satisfying $\Sigma \subseteq \{V, S_N, S_{N-1}, \ldots\}$ for any fixed $N \geq 0$. Recall that an m-primary Σ-path is a lattice path running from $(0, 0)$ to some fixed $(n, -m)$ with $m, n \geq 0$ whose points, except the last one, lie on or above the x-axis. We denote by $\mathcal{P}_\Sigma(n, -m)$ the family of all m-primary Σ-paths running from $(0, 0)$ to $(n, -m)$ (see Definition 1.2 on page 6). An example of a 1-primary Σ-path, where $\Sigma = \{V, U_6, U_5, \ldots, U_0, D_1, D_2\}$, is given in Figure 2.1.

We assume that $\mathcal{P}_\Sigma(0, 0) = \{\lambda\}$, where λ is the empty path with zero steps, and $\mathcal{P}_\Sigma(0, -m)$ is the empty set for $m \geq 1$. For $n = 1$, we have

$$\mathcal{P}_\Sigma(1, -m) = \begin{cases} \{S_k V^{k+m} : S_k \in \Sigma_{\geq -m}\} & \text{if } m \in \{0, 1\} \text{ and } V \in \Sigma, \\ \{S_{-m}\} \cap \Sigma & \text{if } m \in \{0, 1\} \text{ and } V \notin \Sigma, \\ \{S_{-m}\} \cap \Sigma & \text{if } m \geq 2. \end{cases}$$

FIGURE 2.1: A 1-primary path μ running from $(0, 0)$ to $(7, -1)$. Lattice points given by Π_μ are depicted using open circles.

Let $\mu = \mu_1 \cdots \mu_t$ be an m-primary Σ-path in $\mathcal{P}_\Sigma(n, -m)$ such that $m \geq 0$ and $n \geq 2$. Because μ has at least two non-vertical steps, the first step μ_1 is U_h for certain $h \geq 0$. The path runs from $(0, 0)$ to $(m, -n)$; thus, it passes through the points (x_1, h), $(x_2, h-1), \ldots, (x_{h+t}, -m) \in \mathbb{R} \times \mathbb{Z}$ such that they are chosen to be the left-most ones, i.e., $x_i = \min\{x : \mu \text{ passes through } (x, h-i+1)\}$. Note that $x_1 = 1$, and some x_i may not be integers. Let us denote by Π_μ the list of these points such that both coordinates are integers arranged in order from left to right. For instance, all the points of Π_μ for the path μ given in Figure 2.1 are marked by open circles. In the following, we show how Π_μ determines the decomposition of μ.

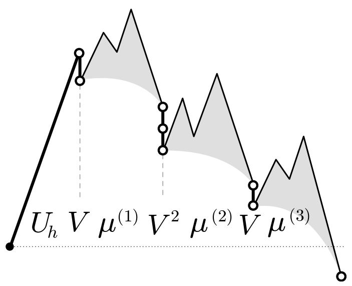

FIGURE 2.2: The decomposition of an m-primary path μ with $m \geq 1$. All points in Π_μ are marked by open circles.

First, suppose that $m \geq 1$ and $\Pi_\mu = (p_1, p_2, \ldots, p_r)$. Note that p_r is the ending point of μ. Cutting μ at points in Π_μ, we obtain $\mu = U_h\, \alpha^{(1)}\, \alpha^{(2)} \cdots \alpha^{(r-1)}$ where $\alpha^{(i)}$ is the subpath of μ between points p_i and p_{i+1} for $1 \leq i \leq r-1$. Moreover, each $\alpha^{(i)}$ is either the vertical step V or a path that contains at least one non-vertical step. Suppose that exactly d of $\alpha^{(1)}, \ldots, \alpha^{(r-1)}$ are not vertical steps and denote them by $\mu^{(1)}, \ldots, \mu^{(d)}$. Observe that for $1 \leq i \leq d$, the ending point of $\mu^{(i)}$ is the first point that lies below the initial one. It follows that $\mu^{(i)}$ is an m_i-primary Σ-path in $\mathcal{P}_\Sigma(n_i, -m_i)$, where m_i is the difference between the y-coordinates of the initial and the ending points of $\mu^{(i)}$, and n_i is the number of non-vertical steps in $\mu^{(i)}$. Note that $m_i \geq 1$ and $n_i \geq 1$. Observe that $\mu^{(i)}$ may contain vertical steps if $V \in \Sigma$, however, it starts with non-vertical step.

Hence, μ can be rewritten as

$$\mu = U_h V^{c_1}\,\mu^{(1)}\,V^{c_2}\,\mu^{(2)}\cdots V^{c_d}\,\mu^{(d)}\,V^{c_{d+1}}. \tag{2.1}$$

where $1 \le d \le h+1$, and for $1 \le i \le d$, we have $\mu_i \in \mathcal{P}_\Sigma(n_i, -m_i)$ with $m_i, n_i \ge 1$, and $c_1, c_2, \ldots, c_{d+1} \ge 0$ (see Figure 2.2 for $d = 3$). Moreover, because the entire path μ is an m-primary path and only the ending point of μ lies below the x-axis, we conclude that the last subpath $\mu^{(d)}$ is an m_d-primary path with $m_d \ge m$. It follows also that if $m \ge 2$, then the last step of μ is a down step and $c_{d+1} = 0$.

Example. Let μ be the path given in Figure 2.3. This path is decomposable as $\mu = U_3 V \mu^{(1)} V \mu^{(2)}$, where $\mu^{(1)} = U_2 U_0 D_1 V U_1 D_2$, $\mu^{(2)} = U_2 D_3$.

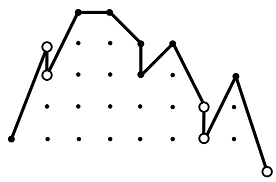

FIGURE 2.3: A 1-primary Σ-path running from $(0,0)$ to $(8,-1)$. Lattice points determining the decomposition of the path are drawn using open circles.

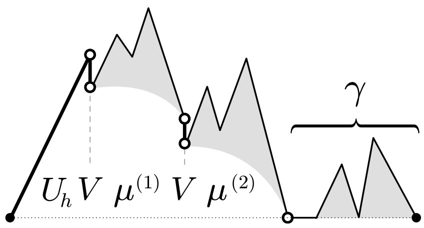

FIGURE 2.4: The decomposition of a 0-primary path π. Points of Π_π are marked by open circles.

Next, suppose that $m = 0$ and $\Pi_\mu = (p_1, p_2, \ldots, p_r)$. Note that p_r may not be the ending point of μ. In fact, p_r is the first point after the first step of μ in which μ touches the x-axis (see Figure 2.4). Let us denote by γ the subpath of μ between p_r and the ending point of μ. An analysis similar to that in the case $m \ge 1$ shows that cutting μ at the points $p_1, p_2, \ldots, p_r$ in Π_μ, we obtain

$$\mu = U_h V^{c_1}\,\mu^{(1)}\,V^{c_2}\,\mu^{(2)}\cdots V^{c_d}\,\mu^{(d)}\,V^{c_{d+1}}\,\gamma, \tag{2.2}$$

for certain d such that $0 \le d \le h$, and for $1 \le i \le d$, we have $\mu^{(i)} \in \mathcal{P}_\Sigma(n_i, -m_i)$ with $m_i, n_i \ge 1$, we have $\gamma \in \mathcal{P}_\Sigma(n_{d+1}, 0)$ with $n_{d+1} \ge 0$, and $c_1, \ldots, c_{d+1} \ge 0$ (see Figure 2.4 for $d = 2$).

Definition 2.1. Let $\mu \in \mathcal{P}_\Sigma(n, -m)$. If $n \ge 2$, then the first step of μ is U_h for certain $h \ge 0$ and μ is decomposable as in (2.1) (for $m \ge 1$) or in (2.2) (for $m = 0$). The *shape* of the path μ is the triple (m, d, k), where $k = h - c_1 - c_2 - \cdots - c_{d+1}$. Additionally, for $n = 1$ and $m \in \{0, 1\}$, we define the shape of $U_h V^{h+m} \in \mathcal{P}_\Sigma(1, -m)$ to be $(m, 0, 0)$.

Example. Let μ be the path given in Figure 2.5. The path is decomposable as $\mu = U_3 \mu^{(1)} V \mu^{(2)} \gamma$, where $\mu^{(1)} = D_1$, $\mu^{(2)} = U_1 U_0 D_1 V$, and $\gamma = U_0 U_2 D_1 D_1$. The shape of μ is $(0, 2, 2)$. The shape of the path given in Figure 2.3 is $(1, 2, 1)$.

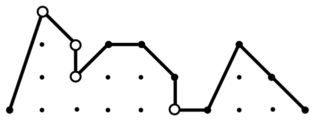

FIGURE 2.5: A 0-primary Σ-path running from $(0, 0)$ to $(9, 0)$. Lattice points which determine the decomposition of the path are drawn using open circles.

Note that if the set of steps Σ does not contain the vertical step V, then the decomposition of an m-primary Σ-path μ, with $m \geq 1$, is given by

$$\mu = U_k \, \mu^{(1)} \mu^{(2)} \cdots \mu^{(d)}, \tag{2.3}$$

where each $\mu^{(i)}$ is in $\mathcal{P}_\Sigma(n_i, -m_i)$ for certain $m_i, n_i \geq 1$ (see Figure 2.6 for $d = 3$). If $m = 0$, then the decomposition of a 0-primary Σ-path μ is given by

$$\mu = U_k \, \mu^{(1)} \mu^{(2)} \cdots \mu^{(d)} \gamma, \tag{2.4}$$

where each $\mu^{(i)}$ is in $\mathcal{P}_\Sigma(n_i, -m_i)$ for certain $m_i, n_i \geq 1$, and $\gamma \in \mathcal{P}_\Sigma(n_{d+1}, 0)$ with $n_{d+1} \geq 0$. It is worth noting that if $\mu^{(i)}$ has at least two non-vertical steps, then it is again decomposable into shorter subpaths.

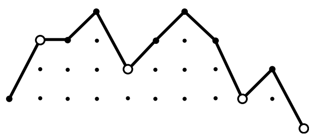

FIGURE 2.6: A 1-primary Σ-path running from $(0, 0)$ to $(10, -1)$ that does not contain vertical steps. Lattice points that determine the decomposition of the path are drawn using open circles.

Lemma 2.2. *Let $\mu = \mu_1 \mu_2 \cdots \mu_t \in \mathcal{P}_\Sigma(n, -m)$, with $m \geq 0$ and $n \geq 1$. Suppose that $\mu_i = U_h$, with $h \geq 0$, and μ_i connects two lattice points (j, l) and $(j+1, l+h)$ for some j and l such that $0 \leq j \leq n-1$ and $l \geq 0$.*

(i) If $m > 0$ or $l \neq 0$, then μ_i uniquely determines the nonempty p-primary Σ-subpath of μ in which μ_i is the first step and $p \geq 1$.

(ii) If $m = 0$ and $l = 0$, then μ_i uniquely determines the shortest nonempty 0-primary Σ-subpath of μ in which μ_i is the first step.

Proof. (*i*) Recall that μ is the path running from $(0,0)$ to $(n,-m)$. It follows that there is at least one point $(x,y) \in \mathbb{Z} \times \mathbb{Z}$ in μ such that $x > j$ and $y < l$. Suppose that (x,y) is the first such point with the minimal first coordinate (see Figure 2.7). Assume that $\mu' = \mu_i \mu_{i+1} \cdots \mu_{i+r}$ is the subpath of μ running from (j,l) to (x,y). Because the ending point of μ' is the first point that lies below the initial point, we conclude that μ' is the only p-primary subpath, with $p \geq 1$, in which μ_i is the first step.

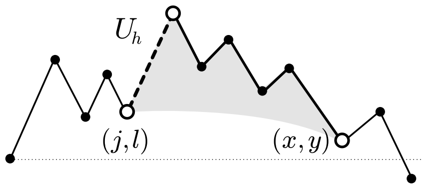

FIGURE 2.7: An up step in an m-primary path with $m \geq 1$.

(*ii*) If $m = 0$ and $l = 0$, then there is at least one point $(x,0)$ in μ such that $x > j$. Suppose that $(x,0)$ is the first such point with the minimal first coordinate (see Figure 2.8). Assume that $\mu' = \mu_i \mu_{i+1} \cdots \mu_{i+r}$ is the subpath of μ running from (j,l) to $(x,0)$. Because the initial and ending points of μ' are the only two points that lie on the x-axis and because there are no points below the x-axis, we conclude that μ' is the shortest 0-primary subpath in which μ_i is the first step. $\square$

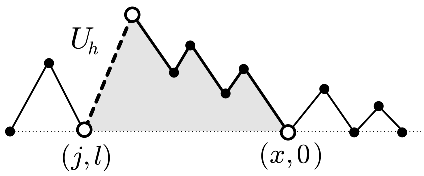

FIGURE 2.8: An up step in a 0-primary path that starts at the x-axis.

2.2 Weighted primary paths

Recall that $\Lambda \subseteq \{V, S_N, S_{N-1}, \ldots\}$ such that $S_N, V \in \Lambda$ for fixed $N \geq 0$. We set

$$\Gamma = (\Lambda \setminus \{V\}) \cup \{U_N, U_{N-1}, \ldots, U_0, D_1\}. \tag{2.5}$$

Note that even though the set Λ does not contain all up steps between U_0 and U_N, the set Γ does. Recall that $\Lambda_{\geq k}$ denotes the set $\{S_h \in \Lambda : h \geq k\}$ for $k \in \mathbb{Z}$.

Definition 2.3. For $m, d \geq 0$ and $0 \leq k \leq N$, let

$$\mathcal{H}_\Lambda(m,d,k) = \Big\{(h,c) : U_h \in \Lambda_{\geq k}, c \in \mathcal{C}(m,d,h-k)\Big\},$$

where $\mathcal{C}(m,d,j)$ is the set of compositions of the number j into $d+1$ parts defined as follows:

$$\mathcal{C}(m,d,j) = \Big\{(c_1, c_2 \ldots, c_{d+1}) : \sum_{i=1}^{d+1} c_i = j,\ c_i \geq 0, \text{ and } c_{d+1} = 0 \text{ if } m \geq 2\Big\}.$$

Definition 2.4. Let $n \geq 1$, $m \geq 0$, and $\mu = \mu_1 \mu_2 \cdots \mu_n \in \mathcal{P}_\Gamma(n,-m)$. Let w_μ be the function $w_\mu : \{1,2,\ldots,n\} \to \mathbb{Z}$ defined as follows. For every $i \in \{1,\ldots,n\}$,
if $\mu_i = D_p$, then we set

$$w_\mu(i) = \begin{cases} |\Lambda_{\geq -1}| & \text{if } p = 1, \\ 1 & \text{if } p \geq 2, \end{cases} \tag{2.6}$$

if $\mu_i = U_h$, then we set

$$w_\mu(i) = |\mathcal{H}_\Lambda(p,d,k)|, \tag{2.7}$$

where (p,d,k) is the shape of the shortest 0-primary subpath of $\mu \in \mathcal{P}_\Gamma(n,0)$ in which μ_i is the first step if μ_i starts at the x-axis and $m=0$ (see Lemma 2.2 (ii)); otherwise, (p,d,k) is the shape of the unique primary subpath of μ in which μ_i is the first step (see Lemma 2.2 (i)). We call $w_\mu(i)$ the *maximal weight* of μ_i. The *weight* of μ, denoted by $w(\mu)$, is the product $w_\mu(1)w_\mu(2)\cdots w_\mu(n)$.

Definition 2.5. A *weighted m-primary Γ-path* is a pair (μ, v) such that

(i) μ is an m-primary Γ-path in $\mathcal{P}_\Gamma(n,-m)$, with $m \geq 0$ and $n \geq 1$, and

(ii) v is a sequence of integers $v_1,\ldots,v_n$ such that $1 \leq v_i \leq w_\mu(i)$ for $i \in \{1,\ldots,n\}$.

For $m \geq 0$ and $n \geq 1$, let $\mathcal{W}_\Gamma^\Lambda(n,-m)$ denote the set of all weighted m-primary Γ-paths in $\mathcal{P}_\Gamma(n,-m)$. We assume that $\mathcal{W}_\Gamma^\Lambda(0,0) = \{\lambda\}$, and $\mathcal{W}_\Gamma^\Lambda(0,-m) = \emptyset$ for $m \geq 1$.

For all $m \geq 0$ and $n \geq 1$, we have

$$|\mathcal{W}_\Gamma^\Lambda(n,-m)| = \sum_{\mu \in \mathcal{P}_\Gamma(n,-m)} w(\mu).$$

Remark. Note that if we look at the value of $w_\mu(i)$ as the number of ways to color the ith step of μ, we see that $w(\mu)$ is the number of ways to color the entire path $\mu \in \mathcal{P}_\Gamma(n,-m)$ according to the weight function w.

Proposition 2.6. *For all $m, d \geq 0$ and $0 \leq k \leq N$, we have*

$$|\mathcal{H}_\Lambda(m,d,k)| = \sum_{U_h \in \Lambda_{\geq k}} \binom{h-k+d-\epsilon_m}{h-k}, \tag{2.8}$$

where $\epsilon_m = 0$ *if* $m \in \{0, 1\}$ *and* $\epsilon_m = 1$ *if* $m \geq 2$.

Proof. Recall that N is the maximal integer h such that $U_h \in \Lambda$. Let us partition the set $\mathcal{H}_\Lambda(m, d, k)$ into pairwise disjoint classes $A_k, A_{k+1}, \dots, A_N$ such that A_h contains such pairs whose first element is h. If $U_h \notin \Lambda_{\geq k}$, then A_h is empty. If $U_h \in \Lambda_{\geq k}$, then the size of A_h is the number of compositions of $h - k$ into $d + 1$ parts for $m \in \{0, 1\}$ or into d parts for $m \geq 2$. In both cases, zero parts are allowed. Therefore, $|A_h| = \binom{h-k+d-\epsilon_m}{h-k}$, and the results follow. □

Corollary 2.7. *If* $\{U_N, U_{N-1}, \dots, U_0\} \subset \Lambda$, *then for all* $d \geq 0$ *and* $0 \leq k \leq N$, *we have*

$$|\mathcal{H}_\Lambda(m, d, k)| = \begin{cases} \binom{N-k+d+1}{d+1} & \text{if } m \in \{0, 1\}, \\ \binom{N-k+d}{d} & \text{if } m \geq 2. \end{cases} \tag{2.9}$$

Proof. Let us consider the size of $\mathcal{H}_\Lambda(m, d, k)$ for $m \in \{0, 1\}$. By Proposition 2.6 and using the properties of the binomial coefficients, we obtain

$$\sum_{U_h \in \Lambda_{\geq k}} \binom{h-k+d-0}{h-k} = \sum_{h=k}^{N} \binom{h-k+d}{h-k} = \binom{N+d-k+1}{d+1}.$$

We show the second formula in the same manner. □

Example. Let $\Lambda = \{U_2, U_1, U_0, D_1, V\}$ and $\Gamma = \Lambda \setminus \{V\}$. Let $(\mu, v) \in \mathcal{W}_\Gamma^\Lambda(8, -1)$ such that $\mu = U_2 D_1 U_1 U_0 D_1 D_1 U_0 D_1$ and $v = (v_1, \dots, v_8)$. Let us calculate the range of each v_i. We have four down steps D_1, i.e., $\mu_2, \mu_5, \mu_6, \mu_8$, and four up steps $\mu_1, \mu_3, \mu_4, \mu_7$. If $\mu_i = D_1$, then $1 \leq v_i \leq |\Lambda_{\geq -1}| = 4$. If μ_i is an up step, then we calculate the shortest primary subpath in which μ_i is the first step (see Lemma 2.2). Having the shape of this subpath, we apply Proposition 2.6 to obtain the maximal value for v_i. We have

$$\begin{aligned} 1 \leq v_1 \leq |\mathcal{H}_\Lambda(1, 3, 2)| = 1, &\quad 1 \leq v_3 \leq |\mathcal{H}_\Lambda(1, 2, 1)| = 4, \\ 1 \leq v_4, v_7 \leq |\mathcal{H}_\Lambda(1, 1, 0)| = 6, &\quad 1 \leq v_2, v_5, v_6, v_8 \leq |\Lambda_{\geq -1}| = 4. \end{aligned}$$

The weight of μ is $w(\mu) = w_\mu(1) \cdots w_\mu(n) = 36864$.

There are several examples of weighted Σ-paths in the literature. Two of the most well-known are the weighted Motzkin and weighted Lukasiewicz paths mentioned in Section 1. Recall that a (w_1, w_2, w_3)-Motzkin path is a Motzkin path in which there are w_k types of the step S_{k-2} for $k \in \{1, 2, 3\}$.

Example. If $\Lambda = \{U_1, U_0, V\}$ and $\Gamma = \{U_1, U_0, D_1\}$, then the paths of $\mathcal{W}_\Gamma^\Lambda(n, -1)$ are $(2, 3, 1)$-Motzkin paths considered by Chen and Wang [39]. These paths run from $(0, 0)$ to $(n, -1)$, where each step D_1 has two types, each step U_0 has three types, and each step U_1 has one type. Note that in this case, every subpath that starts with U_0 has the shape $(1,1,0)$, and every subpath that starts with U_1 has the shape $(1,2,1)$. In the class $\mathcal{W}_\Gamma^\Lambda(n,0)$, the maximal weights of D_1 and U_1 are 2 and 1, respectively. The maximal weight of U_0 depends on where it starts, namely, whether it starts on the x-axis. For a U_0 that starts on the x-axis, the maximal weight is $|\mathcal{H}_\Lambda(0,0,0)| = 2$, and for a U_0 that starts above the x-axis, the maximal weight is $|\mathcal{H}_\Lambda(1,1,0)| = 3$. This class is exactly the class of the large $(2,3,1)$-Motzkin paths considered by Chen and Wang [39]. For $m \geq 2$, the families $\mathcal{P}_\Gamma(n,-m)$ and $\mathcal{W}_\Gamma^\Lambda(n,-m)$ are empty.

Example. If $\Lambda = \{U_1, V\}$ and $\Gamma = \{U_1, U_0, D_1\}$, then the paths of $\mathcal{W}_\Gamma^\Lambda(n,-1)$ are $(1,2,1)$-Motzkin paths considered by Deutsch and Shapiro [11]. In this case, there are two types of the step U_0, one type of D_1, and one type of U_1.

Lemma 2.8. *Let $m \geq 1$, $n \geq 2$, $d \geq 1$, and $0 \leq k \leq N$. Suppose that for $1 \leq i \leq d$, we have $\mu^{(i)} \in \mathcal{P}_\Gamma(n_i,-m_i)$ and $\pi^{(i)} \in \mathcal{P}_\Lambda(n_i,-m_i)$, with $m_i, n_i \geq 1$, and $m_1 + m_2 + \cdots + m_d - k = m$, and $1 + n_1 + \cdots + n_d = n$. Let $(h,c) \in \mathcal{H}_\Lambda(m,d,k)$, with $c = (c_1, \ldots, c_{d+1})$. We have*

$$\mu = U_k\, \mu^{(1)}\, \mu^{(2)} \cdots \mu^{(d)} \;\in\; \mathcal{P}_\Gamma(n,-m) \tag{2.10}$$

if and only if

$$\pi = U_h\, V^{c_1}\, \pi^{(1)}\, V^{c_2}\, \pi^{(2)} \cdots V^{c_d}\, \pi^{(d)}\, V^{c_{d+1}} \;\in\; \mathcal{P}_\Lambda(n,-m). \tag{2.11}$$

Proof. We first show that μ and π end at the same points (see Figure 2.9). The first step of μ is U_k. The first step of π is U_h such that $h \geq k$, and there is some number of vertical steps between subpaths $\pi^{(i)}$. Let us denote by $(\mathrm{x}(\mu), \mathrm{y}(\mu))$ and $(\mathrm{x}(\pi), \mathrm{y}(\pi))$ the ending points of μ and π, respectively. Observe that π and μ have the same number of non-vertical steps. Indeed, for $1 \leq i \leq d$, the number of non-vertical steps in $\mu^{(i)}$ is that of $\pi^{(i)}$. Thus, $\mathrm{x}(\mu) = \mathrm{x}(\pi)$. Let us now compute

$$\begin{aligned}
\mathrm{y}(\mu) &= k - m_1 - \cdots - m_d,\\
\mathrm{y}(\pi) &= h - m_1 - \cdots - m_d - c_1 - \cdots - c_{d+1}.
\end{aligned}$$

Because (h,c) is the pair in $\mathcal{H}_\Lambda(m,d,k)$ (see Definition 2.3), we have $k = h - c_1 - \cdots - c_{d+1}$. Thus, $\mathrm{y}(\mu) = \mathrm{y}(\pi) = -m$. Recall that $V \in \Lambda$, and h is such an integer that $U_h \in \Lambda$. Thus, the entire path π is a Λ-path. On the other hand, for $0 \leq k \leq N$, we have $U_k \in \Gamma$ (see Definition 2.5); therefore, μ is a Γ-path.

Finally, we must show the following: (i) if μ is a primary path, i.e., only the ending point lies below the x-axis, then so is π, and (ii) if π is a primary path, then so is μ. To

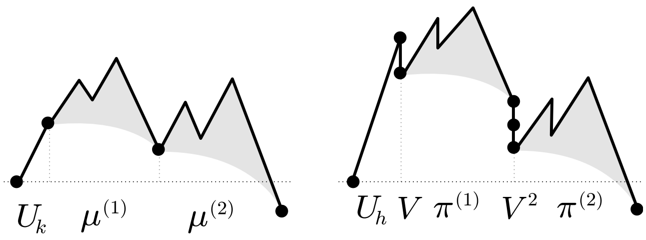

FIGURE 2.9: An illustration of Lemma 2.8 for $d = 2$. The left path is μ given by (2.10), and the right path is π given by (2.11).

see (i), observe that $m_d \geq m$ and $c_{d+1} = 0$ if $m \geq 2$. Moreover, all points except the first one in the subpath $U_h V^{c_1} \pi^{(1)} \cdots V^{c_{d-1}} \pi^{(d-1)} V^{c_d}$ lie weakly above the initial point of $\pi^{(d)}$. Similarly, to see (ii), observe that if $m = 1$, then $m_d \geq 1$, and if $m \geq 2$, then $c_{d+1} = 0$ and $m_d \geq m$. All points except the first one in the subpath $U_k \mu^{(1)} \cdots \mu^{(d-1)}$ lie weakly above the initial point of $\mu^{(d)}$. $\square$

Remark. The above-mentioned lemma is for m-primary paths with $m \geq 1$. We now state the analogue of this result for $m = 0$.

Lemma 2.9. *Let $n \geq 2$, $d \geq 0$, and $0 \leq k \leq N$. Suppose that for $1 \leq i \leq d$, we have $\mu^{(i)} \in \mathcal{P}_\Gamma(n_i, -m_i)$ and $\pi^{(i)} \in \mathcal{P}_\Lambda(n_i, -m_i)$, with $m_i, n_i \geq 1$, and $m_1 + m_2 + \cdots + m_d = k$. Suppose that $\gamma \in \mathcal{P}_\Gamma(n_{d+1}, 0)$, $\gamma' \in \mathcal{P}_\Lambda(n_{d+1}, 0)$, with $n_{d+1} \geq 0$, and $1 + n_1 + \cdots + n_{d+1} = n$. Let $(h, c) \in \mathcal{H}_\Lambda(0, d, k)$, with $c = (c_1, \ldots, c_{d+1})$. We have*

$$\mu = U_k \, \mu^{(1)} \, \mu^{(2)} \cdots \mu^{(d)} \gamma \ \in \ \mathcal{P}_\Gamma(n, 0) \tag{2.12}$$

if and only if

$$\pi = U_h \, V^{c_1} \, \pi^{(1)} \, V^{c_2} \, \pi^{(2)} \cdots V^{c_d} \, \pi^{(d)} \, V^{c_{d+1}} \gamma' \ \in \ \mathcal{P}_\Lambda(n, 0). \tag{2.13}$$

Proof. This can be proved in much the same way as Lemma 2.8. The only difference is due to the presence of the additional subpaths γ and γ' which are possibly empty 0-primary path (see Figure 2.10 for $d = 2$). $\square$

2.3 Bijection between weighted Γ-paths and Λ-paths

Recall that $\Lambda \subseteq \{V, S_N, S_{N-1}, \ldots\}$ such that $S_N, V \in \Lambda$, and $\Gamma = (\Lambda \setminus \{V\}) \cup \{U_N, U_{N-1}, \ldots, U_0, D_1\}$, for fixed $N \geq 0$. Let us denote

$$\mathcal{P}_\Lambda = \bigcup_{m \geq 0} \bigcup_{n \geq 0} \mathcal{P}_\Lambda(n, -m), \qquad \mathcal{W}_\Gamma^\Lambda = \bigcup_{m \geq 0} \bigcup_{n \geq 0} \mathcal{W}_\Gamma^\Lambda(n, -m).$$

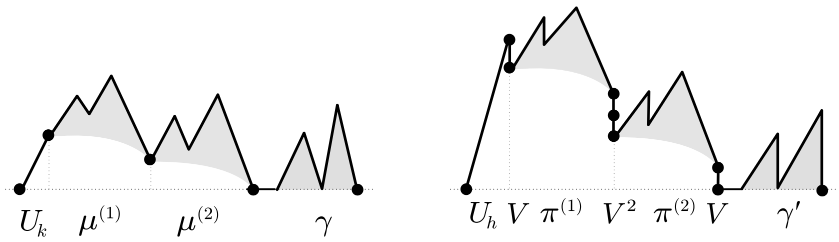

FIGURE 2.10: An illustration of Lemma 2.9 for $d = 2$. The left path is μ given by (2.12), and the right path is π given by (2.13).

Theorem 2.10. *For all $m \geq 0$ and $n \geq 0$, we have*

$$|\mathcal{W}_\Gamma^\Lambda(n,-m)| = |\mathcal{P}_\Lambda(n,-m)|.$$

Proof. To prove the assertion, we shall

- define a map $f : \mathcal{W}_\Gamma^\Lambda \to \mathcal{P}_\Lambda$ (Definition 2.11 on page 22),
- define a map $g : \mathcal{P}_\Lambda \to \mathcal{W}_\Gamma^\Lambda$ (Definition 2.13 on page 25), and
- prove that for all $m \geq 0$ and $n \geq 0$, the map $f : \mathcal{W}_\Gamma^\Lambda(n,-m) \to \mathcal{P}_\Lambda(n,-m)$ is the inverse function of $g : \mathcal{P}_\Lambda(n,-m) \to \mathcal{W}_\Gamma^\Lambda(n,-m)$ (Lemma 2.15 on page 28).

□

Definition 2.11. The map $f : \mathcal{W}_\Gamma^\Lambda \to \mathcal{P}_\Lambda$.

The definition is recursive. Let (μ, v) be a weighted path in $\mathcal{W}_\Gamma^\Lambda(n,-m)$, with $v = (v_1, \ldots, v_n)$. For $n = 0$, we have $\mathcal{W}_\Gamma^\Lambda(0,0) = \{\lambda\}$, and we set $f(\lambda) = \lambda$. If $n = 1$ and $m = 0$, then $\mu = U_0$, and $1 \leq v_1 \leq |\mathcal{H}_\Lambda(0,0,0)|$. Suppose that (h,c) is the v_1th pair in $\mathcal{H}_\Lambda(0,0,0)$. Note that $c = (h)$. We set

$$f((U_0, v)) = U_h V^h. \tag{2.14}$$

If $n = 1$ and $m = 1$, then $\mu = D_1$, and $1 \leq v_1 \leq |\Lambda_{\geq -1}|$. Suppose that S_h is the v_1th step in $\Lambda_{\geq -1}$. We set

$$f((D_1, v)) = S_h V^{h+1}. \tag{2.15}$$

If $n = 1$ and $m \geq 2$, then $\mu = D_m$ and $v_1 = 1$. We set

$$f((D_m, v)) = D_m. \tag{2.16}$$

For $n \geq 2$ and $m \geq 1$, suppose that the first step of μ is an up step U_k and that the entire path can be decomposed as follows (see Section 2.1 for more details):

$$\mu = U_k\, \mu^{(1)} \mu^{(2)} \cdots \mu^{(d)}, \tag{2.17}$$

where each $\mu^{(i)}$ is in $\mathcal{P}_\Gamma(n_i, -m_i)$ for certain $m_i, n_i \geq 1$. The shape of μ (see Definition 2.1) is (m, d, k). Let us decompose the sequence of weights $v = (v_1, \ldots, v_n)$ into v_1 and d subsequences $v^{(1)}, v^{(2)}, \ldots, v^{(d)}$ of consecutive elements of v according to the lengths of $\mu^{(1)}, \ldots, \mu^{(d)}$. Specifically,

$$v = (v_1, \underbrace{v_2, \ldots, v_{s(1)}}_{v^{(1)}}, \underbrace{v_{s(1)+1}, \ldots, v_{s(2)}}_{v^{(2)}}, \cdots \underbrace{v_{s(d-1)+1}, \ldots, v_{s(d)}}_{v^{(d)}}), \tag{2.18}$$

where $s(i) = 1 + n_1 + n_2 + \cdots + n_i$ for $1 \leq i \leq d$. The weight of the first step is v_1, which is an integer in $\{1, 2, \ldots, |\mathcal{H}_\Lambda(m, d, k)|\}$ (see (2.7)). Suppose that (h, c) is the v_1th pair in $\mathcal{H}_\Lambda(m, d, k)$, with $c = (c_1, c_2, \ldots, c_{d+1})$. We set

$$f((\mu, v)) = U_h\, V^{c_1}\, \pi^{(1)}\, V^{c_2}\, \mu^{(2)} \cdots V^{c_d}\, \pi^{(d)}\, V^{c_{d+1}}, \tag{2.19}$$

where $\pi^{(i)} = f((\mu^{(i)}, v^{(i)}))$ for $1 \leq i \leq d$ (see Figure 2.11 for $d = 3$).

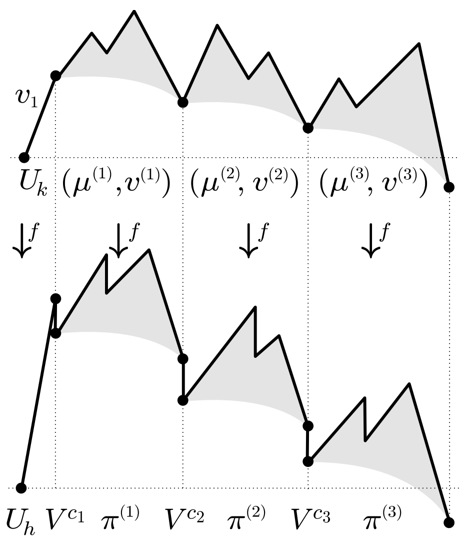

FIGURE 2.11: Action of the function f on a weighted m-primary Γ-path with $m \geq 1$. The function f changes the first up step U_k into U_h, adds $h - k$ vertical steps between subpaths, and changes each $(\mu^{(i)}, v^{(i)})$ into $\pi^{(i)}$.

Similarly, for $n \geq 2$ and $m = 0$, suppose that μ can be decomposed as follows:

$$\mu = U_k\, \mu^{(1)} \mu^{(2)} \cdots \mu^{(d)}\, \gamma, \tag{2.20}$$

where each $\mu^{(i)}$ is in $\mathcal{P}_\Gamma(n_i, -m_i)$ for certain $m_i, n_i \geq 1$, and γ is a possibly empty 0-primary path in $\mathcal{P}_\Gamma(n_{d+1}, 0)$ (see Section 2.1). The shape of μ is $(0, d, k)$. As in the case $m \geq 1$, we decompose the sequence of weights v into v_1 and $d + 1$ subsequences $v^{(1)}, \ldots, v^{(d+1)}$ according to the lengths of $\mu^{(1)}, \ldots, \mu^{(d)}$, and γ in the same way as in (2.18). Note that if γ is the empty path, then $v^{(d+1)}$ is the empty sequence. The weight of the first step is $v_1 \in \{1, 2, \ldots, |\mathcal{H}_\Lambda(m, d, k)|\}$ (see (2.7)). Suppose that (h, c) is the v_1th pair in $\mathcal{H}_\Lambda(m, d, k)$ and $c = (c_1, c_2, \ldots, c_{d+1})$. We set

$$f((\mu, v)) = U_h\, V^{c_1}\, \pi^{(1)}\, V^{c_2}\, \pi^{(2)} \cdots V^{c_d}\, \pi^{(d)}\, V^{c_{d+1}}\, \gamma', \tag{2.21}$$

where $\pi^{(i)} = f((\mu^{(i)}, v^{(i)}))$ for $1 \le i \le d$ and $\gamma' = f(\gamma, v^{(d+1)})$ (see Figure 2.12 for $d = 2$).

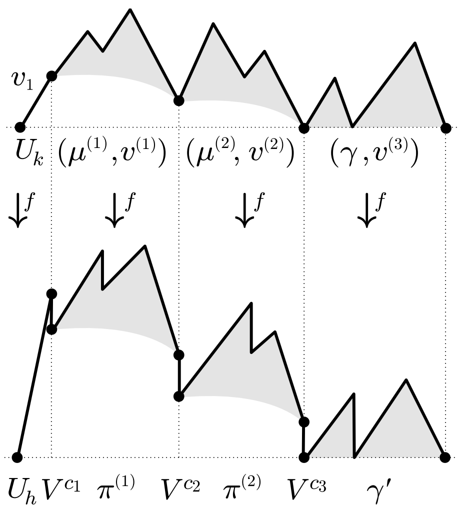

FIGURE 2.12: Action of the function f on a weighted 0-primary Γ-path. The function f changes the first up step U_k into U_h, adds $h - k$ vertical steps between subpaths, changes each $(\mu^{(i)}, v^{(i)})$ into $\mu^{(i)}$, and changes $(\gamma, v^{(d+1)})$ into γ'.

Lemma 2.12. *For all $m \ge 0$ and $n \ge 1$, if $(\mu, v) \in \mathcal{W}_\Gamma^\Lambda(n, -m)$, then $f((\mu, v)) \in \mathcal{P}_\Lambda(n, -m)$.*

Proof. The proof is by induction on n. Let $\pi = f((\mu, v))$. If $n = 1$, then we have $\mu = S_m$. By (2.14), (2.15), and (2.16), we see that $\pi \in \mathcal{P}_\Lambda(1, -m)$.

For $n \ge 2$, suppose that π is given by (2.19) for $m \ge 1$ or by (2.21) for $m = 0$. Recall that for $1 \le i \le d$, the path $(\mu^{(i)}, v^{(i)})$ is in $\mathcal{W}_\Gamma^\Lambda(n_i, -m_i)$. Additionally, if $m = 0$, then $(\gamma, v^{(d+1)})$ is in $\mathcal{W}_\Gamma^\Lambda(n_{d+1}, 0)$. By the induction hypothesis, for $1 \le i \le d$, the subpath $\pi^{(i)} = f((\mu^{(i)}, v^{(i)}))$ is an m_i-primary path in $\mathcal{P}_\Lambda(n_i, -m_i)$, and if $m = 0$, then $\gamma' = f((\gamma, v^{(d+1)}))$ is a 0-primary path in $\mathcal{P}_\Lambda(n_{d+1}, 0)$. Recall that μ is an m-primary path in $\mathcal{P}_\Gamma(n, -m)$. The weight v_1 of the first step of μ is an integer in the set $\{1, 2, \ldots, |\mathcal{H}_\Lambda(m, d, k)|\}$, where (m, d, k) is the shape of μ. Therefore, applying Lemma 2.8 for $m \ge 1$ or Lemma 2.9 for $m = 0$, we see that the entire path π is an m-primary Λ-path in $\mathcal{P}_\Lambda(n, -m)$. □

Example. Let $\Lambda = \{V, U_6, U_5, \ldots, U_0, D_1, D_2\}$ and $\Gamma = \Lambda \setminus \{V\}$. Take $(\mu, v) \in \mathcal{W}_\Gamma^\Lambda(7, -1)$, where $\mu = U_4 D_2 U_0 D_1 U_1 D_2 D_1$ (see the left-hand side of Figure 2.13) and

$v = (v_1, \ldots, v_7)$. The path is decomposable as $\mu = U_4\, \mu^{(1)}\mu^{(2)}\mu^{(3)}\mu^{(4)}$, where

$$\begin{array}{llll} \mu^{(1)} = D_2, & \mu^{(2)} = U_0 D_1, & \mu^{(3)} = U_1 D_2, & \mu^{(4)} = D_1, \\ v^{(1)} = (v_2), & v^{(2)} = (v_3, v_4), & v^{(3)} = (v_5, v_6), & v^{(4)} = (v_7). \end{array}$$

Thus, the shape of the path μ is $(1, 4, 4)$. By (2.7), $v_1 \in \{1, 2, \ldots, |\mathcal{H}_\Lambda(1, 4, 4)|\}$, where $|\mathcal{H}_\Lambda(1, 4, 4)| = 21$ (see Corollary 2.7). Suppose that the v_1th pair in the set $\mathcal{H}_\Lambda(1, 4, 4)$ is $(6, c)$, where $c = (0, 0, 1, 1, 0)$ is the composition of 2 into 5 parts. Thus,

$$f((\mu, v)) = U_6 V^0 \pi^{(1)} V^0 \pi^{(2)} V^1 \pi^{(3)} V^1 \pi^{(4)} V^0,$$

where $\pi^{(i)} = f(\mu^{(i)}, v^{(i)})$ for $i \in \{1, 2, 3, 4\}$. If we suppose that

$$\pi^{(1)} = D_2, \qquad \pi^{(2)} = U_1 V D_1, \qquad \pi^{(3)} = U_2 D_2 V, \qquad \pi^{(4)} = U_1 V^2,$$

then the final path $f((\mu, v))$ is given on the right-hand side of Figure 2.13.

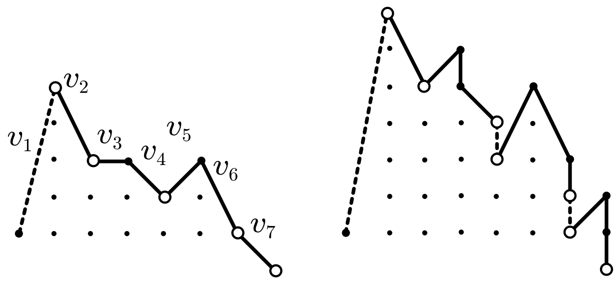

FIGURE 2.13: A weighted Γ-path (left) of $\mathcal{W}_\Gamma^\Lambda(7, -1)$ and corresponding Λ-path (right) of $\mathcal{P}_\Lambda(1, 7)$ under the function f.

Definition 2.13. The map $g : \mathcal{P}_\Lambda \to \mathcal{W}_\Gamma^\Lambda$.

The definition is recursive. Let π be a path in $\mathcal{P}_\Lambda(n, -m)$. For $n = 0$, we have $\mathcal{P}_\Lambda(0, 0) = \{\lambda\}$, and we set $g(\lambda) = \lambda$. If $n = 1$ and $m = 0$, then $\pi = U_h V^h$ for certain $U_h \in \Lambda$. Suppose that the pair $(h, (h))$ is the v_1th pair in $\mathcal{H}_\Lambda(0, 0, 0)$. We set

$$g(U_h V^h) = (U_0, (v_1)). \tag{2.22}$$

If $n = 1$ and $m = 1$, then $\pi = S_h V^{h+1}$ for certain $S_h \in \Lambda_{\geq -1}$. Suppose that S_h is the v_1th step in $\Lambda_{\geq -1}$. We set

$$g(U_h V^{h+1}) = (D_1, (v_1)). \tag{2.23}$$

If $n = 1$ and $m \geq 2$, then $\pi = D_m$. We set

$$g(D_m) = (D_m, (1)). \tag{2.24}$$

For $n \geq 2$ and $m \geq 1$, suppose that the first step of π is an up step U_h and that the entire path π can be decomposed as follows (see Section 2.1 for more details):

$$\pi = U_h V^{c_1} \pi^{(1)} V^{c_1} \pi^{(2)} \cdots V^{c_d} \pi^{(d)} V^{c_{d+1}}, \tag{2.25}$$

where each $\pi^{(i)}$ is in $\mathcal{P}_\Lambda(n_i, -m_i)$ for certain $m_i, n_i \geq 1$. The shape of π is (m, d, k), where $k = h - c_1 - \cdots - c_{d+1}$. Let $c = (c_1, \ldots, c_{d+1})$ and suppose that (h, c) is the v_1th pair in $\mathcal{H}_\Lambda(m, d, k)$. Let $g(\pi^{(i)}) = (\mu^{(i)}, v^{(i)})$ for $i \in \{1, \ldots, d\}$. We set

$$g(\pi) = \left(U_k\, \mu^{(1)}\, \mu^{(2)} \cdots \mu^{(d)}, v\right), \tag{2.26}$$

where $v = (v_1, v_2, \ldots, v_n)$ is the concatenation of v_1, $v^{(1)}, \ldots, v^{(d-1)}$, and $v^{(d)}$ (see Figure 2.14 for $d = 3$).

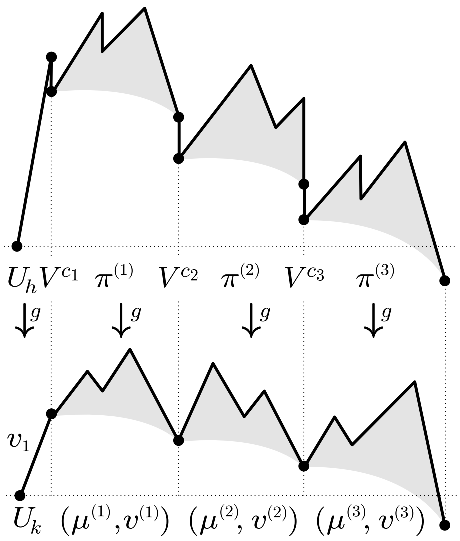

FIGURE 2.14: Action of the function g on an m-primary Λ-path. The function g changes the first up step U_h into U_k, removes all $h - k$ vertical steps between subpaths $\pi^{(i)}$, and changes each $\pi^{(i)}$ into $(\mu^{(i)}, v^{(i)})$.

Similarly, for $n \geq 2$ and $m = 0$, suppose that π can be decomposed as follows,

$$\pi = U_h V^{c_1} \pi^{(1)} V^{c_1} \pi^{(2)} \cdots V^{c_d} \pi^{(d)} V^{c_{d+1}} \gamma', \tag{2.27}$$

where each $\pi^{(i)}$ is in $\mathcal{P}_\Lambda(n_i, -m_i)$ for certain $m_i, n_i \geq 1$, and $\gamma \in \mathcal{P}_\Lambda(n_{d+1}, 0)$ for certain $m_{d+1} \geq 0$. The shape of π is (m, d, k), where $k = h - c_1 - \cdots - c_{d+1}$. Let $c = (c_1, \ldots, c_{d+1})$ and suppose that (h, c) is the v_1th pair in $\mathcal{H}_\Lambda(m, d, k)$. Let $g(\gamma') = (\gamma, v^{(d+1)})$, and $g(\pi^{(i)}) = (\mu^{(i)}, v^{(i)})$ for $i \in \{1, \ldots, d\}$. We set

$$g(\pi) = \left(U_k\, \mu^{(1)}\, \mu^{(2)} \cdots \mu^{(d)}\, \gamma, v\right), \tag{2.28}$$

where $v = (v_1, v_2, \ldots, v_n)$ is the concatenation $v_1\, v^{(1)} v^{(2)} \cdots v^{(d+1)}$ (see Figure 2.15 for $d = 2$). Note that if γ' is the empty path, then $v^{(d+1)}$ is the empty sequence.

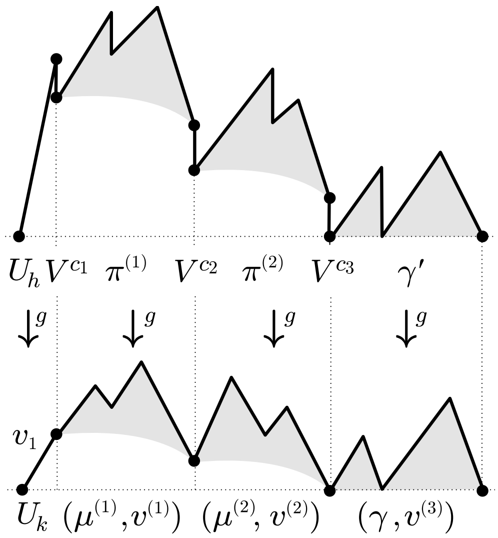

FIGURE 2.15: Action of the function g on a 0-primary Λ-path. The function g changes the first up step U_h into U_k, removes all $h - k$ vertical steps between subpaths $\pi^{(i)}$, changes each $\pi^{(i)}$ into $(\mu^{(i)}, v^{(i)})$, and changes γ' into $(\mu, v^{(d+1)})$.

Example. As in the previous example, let $\Lambda = \{V, S_6, S_5, \ldots S_{-2}\}$ and $\Gamma = \Lambda \setminus \{V\}$. Let $\pi \in \mathcal{P}_\Lambda(7, -1)$ be the path given on the right-hand side of Figure 2.13. The path π is decomposable as $U_6\, \pi^{(1)} \pi^{(2)} V \pi^{(3)} V \pi^{(4)}$, where

$$\pi^{(1)} = D_2, \quad \pi^{(2)} = U_1 V D_1, \quad \pi^{(3)} = U_2 D_2 V, \quad \pi^{(4)} = U_1 V^2.$$

The shape of π is $(1, 4, 4)$ and $c = (0, 0, 1, 1, 0)$. Let $g(\pi^{(i)}) = (\mu^{(i)}, v^{(i)})$ for $i \in \{1, 2, 3, 4\}$, and suppose that

$$\begin{aligned} &\mu^{(1)} = D_2, && \mu^{(2)} = U_0 D_1, && \mu^{(3)} = U_1 D_2, && \mu^{(4)} = D_1, \\ &v^{(1)} = (v_2), && v^{(2)} = (v_3, v_4), && v^{(3)} = (v_5, v_6), && v^{(4)} = (v_7). \end{aligned}$$

In Example 2.3, we have assumed that the pair $(6, c)$ is the v_1th element in $\mathcal{H}_\Lambda(1, 4, 4)$. Thus $g(\pi) = (\mu, v)$, where $v = v_1 v^{(1)} v^{(2)} v^{(3)} v^{(4)}$ and $\mu = U_4\, \mu^{(1)}\, \mu^{(2)} \cdots \mu^{(4)}$. The resulting path μ is given on the left-hand side of Figure 2.13.

Lemma 2.14. *For all $m \geq 0$ and $n \geq 1$, if $\pi \in \mathcal{P}_\Lambda(n, -m)$, then $g(\pi) \in \mathcal{W}_\Gamma^\Lambda(n, -m)$.*

Proof. The proof is by induction on n and goes in much the same way as the proof of Lemma 2.12. Let $(\mu, v) = g(\pi)$. If $n = 1$, then we have $\pi = S_h V^{h+m}$ for $m \in \{0, 1\}$ or $\pi = D_m$ for $m \geq 2$. By (2.22), (2.23), and (2.24), we see that $(\mu, v) \in \mathcal{W}_\Gamma^\Lambda(1, -m)$.

For $n \geq 2$, suppose that the weighted path (μ, v) is given by (2.26) for $m \geq 1$ or by (2.28) for $m = 0$. Recall that for $1 \leq i \leq d$, the path $\pi^{(i)}$ is in $\mathcal{P}_\Lambda(n_i, -m_i)$. Additionally, if $m = 0$, then $\gamma' \in \mathcal{P}_\Lambda(n_{d+1}, 0)$. By the induction hypothesis, for $1 \leq i \leq d$, the subpath $(\mu^{(i)}, v^{(i)}) = g(\pi^{(i)})$ is a weighted m_i-primary path in $\mathcal{W}_\Gamma^\Lambda(n_i, -m_i)$. Additionally, if $m = 0$, then $(\gamma, v^{(d+1)}) = g(\gamma')$ is a weighted 0-primary path in $\mathcal{W}_\Gamma^\Lambda(n_{d+1}, 0)$. Recall that π is an m-primary path in $\mathcal{P}_\Lambda(n, -m)$ and the pair (h, c) belongs to $\mathcal{H}_\Lambda(m, d, k)$, where (m, d, k) is the shape of π. Therefore, applying Lemma 2.8 for $m \geq 1$ or Lemma 2.9 for $m = 0$, we see that the entire path (μ, v) is a weighted m-primary Γ-path in $\mathcal{W}_\Gamma^\Lambda(n, -m)$. $\square$

Lemma 2.15. *We have $f^{-1} = g$.*

Proof. The proof is by induction on n. For $n = 0$, we have $\mathcal{P}_\Lambda(0, 0) = \mathcal{W}_\Gamma^\Lambda(0, 0) = \{\lambda\}$, and $\mathcal{P}_\Lambda(0, -m) = \mathcal{W}_\Gamma^\Lambda(0, -m) = \emptyset$, for $m > 0$.

(i) First, we prove that for every path $(\mu, v) \in \mathcal{W}_\Gamma^\Lambda(n, -m)$, with $m \geq 0$ and $n \geq 1$, we have $g(f((\mu, v))) = (\mu, v)$. Let $\pi = f((\mu, v))$. For $n = 1$, if $m \geq 0$ and $S_{-m} \notin \Gamma$, then $\mathcal{W}_\Gamma^\Lambda(1, -m)$ is empty. If $S_{-m} \in \Gamma$, then $\mu = S_{-m}$ and $v = (v_1)$. If $m = 0$, then $1 \leq v_1 \leq |\mathcal{H}_\Lambda(0, 0, 0)|$. Further, $\pi = U_h V^h$, where $(h, (h))$ is the v_1th pair in the set $\mathcal{H}_\Lambda(0, 0, 0)$. On the other hand, we have $g(\pi) = (U_0, (v_1))$, as claimed. If $m = 1$, then $\mu = D_1$ and $1 \leq v_1 \leq |\Lambda_{\geq -1}|$. Thus, $\pi = S_h V^{h+1}$, where S_h is the v_1th step in $\Lambda_{\geq -1}$. On the other hand, $g(\pi) = (D_1, (v_1)) = \mu$, as claimed. If $m \geq 2$, then $\mu = D_m$ and $v_1 = 1$. Thus, $\pi = D_m$, which implies $g(\pi) = (\mu, v_1)$.

Herein, assume that $n \geq 2$ and suppose that

$$
\begin{aligned}
\mu &= U_k\, \mu^{(1)}\, \mu^{(2)} \cdots \mu^{(d)} && \text{for } m \geq 1, \\
\mu &= U_k\, \mu^{(1)} \mu^{(2)} \cdots \mu^{(d)}\, \gamma && \text{for } m = 0,
\end{aligned}
$$

where $\mu^{(i)} \in \mathcal{P}_\Gamma(n_i, -m_i)$ for certain $m_i, n_i \geq 1$, and γ is a possibly empty path path in $\mathcal{P}_\Gamma(n_{d+1}, 0)$ for certain $n_{d+1} \geq 0$. The shape of μ is (m, d, k) in both cases. We also decompose the sequence of weights v into v_1 and $v^{(1)}, \ldots, v^{(d)}$, and possibly $v^{(d+1)}$ for $m = 0$, according to the lengths of $\mu^{(i)}$ of γ if $m = 0$. Note that $1 \leq v_1 \leq |\mathcal{H}_\Lambda(m, d, k)|$. Assume that (h, c) is the v_1th pair in $\mathcal{H}_\Lambda(m, d, k)$ and $c = (c_1, \ldots, c_{d+1})$. On the one hand, by the definition of f, we have

$$
\begin{aligned}
\pi &= U_h\, V^{c_1}\, \pi^{(1)} V^{c_1} \pi^{(2)} \cdots V^{c_d} \pi^{(d)} V^{c_{d+1}} && \text{for } m \geq 1, \\
\pi &= U_h\, V^{c_1}\, \pi^{(1)} V^{c_1} \pi^{(2)} \cdots V^{c_d} \pi^{(d)} V^{c_{d+1}} \gamma' && \text{for } m = 0,
\end{aligned}
$$

where $\pi^{(i)} = f((\mu^{(i)}, v^{(i)}))$ for $1 \le i \le d$; additionally, $\gamma' = f((\gamma, v^{(d+1)}))$ if $m = 0$. On the other hand, $\pi = f((\mu, v))$ can be uniquely decomposed (see Section 2.1) as

$$\pi = U_h V^{\rho_1} \hat{\pi}^{(1)} V^{\rho_2} \hat{\pi}^{(2)} \cdots V^{\rho_t} \hat{\pi}^{(t)} V^{\rho_{t+1}} \qquad \text{for } m \ge 1,$$
$$\pi = U_h V^{\rho_1} \hat{\pi}^{(1)} V^{\rho_2} \hat{\pi}^{(2)} \cdots V^{\rho_t} \hat{\pi}^{(t)} V^{\rho_{t+1}} \hat{\gamma}' \qquad \text{for } m = 0,$$

for certain primary Λ-paths $\hat{\pi}^{(1)}, \dots, \hat{\pi}^{(t)}$ and a possibly empty 0-primary Λ-path $\hat{\gamma}'$.

Now, we shall show that these two decompositions of π are the same in the sense that $d = t$, for $1 \le i \le d$, we have $c_i = \rho_i$, $\pi^{(i)} = \hat{\pi}^{(i)}$, and $\gamma' = \hat{\gamma}'$. First, observe that $\pi^{(1)} = f((\mu^{(1)}, v^{(1)}))$ is a nonempty m_1-primary path in $\mathcal{P}_\Gamma(n_1, -m_1)$ that starts with a non-vertical step. This implies that $U_h V^{c_1} = U_h V^{\rho_1}$, and $c_1 = \rho_1$. Next, the ending points of $\mu^{(1)}$ and $\hat{\mu}^{(1)}$ are the first points of π that lie below the ending point of $U_h V^{c_1}$. This implies that $\pi^{(1)} = \hat{\pi}^{(1)}$. Continuing in this fashion, we show that $c_i = \rho_i$ and $\pi^{(i)} = \hat{\pi}^{(i)}$ for $1 \le i \le d$, $t = d$, and $\gamma' = \hat{\gamma}'$.

By the induction hypothesis, we see that $g(f((\mu^{(i)}, v^{(i)})) = (\mu^{(i)}, v^{(i)})$ for $1 \le i \le d$. Additionally, if $m = 0$, then $g(f((\gamma, v^{(d+1)})) = (\gamma, v^{(d+1)})$. According to the above assumption that (h, c) is the v_1th pair in $\mathcal{H}_\Lambda(m, d, k)$, the resulting path $g(\pi)$ is

$$g(\pi) = \Big(U_k\, \mu^{(1)} \mu^{(2)} \cdots \mu^{(d)}, v\Big) \qquad \text{for } m \ge 1,$$
$$g(\pi) = \Big(U_k\, \mu^{(1)} \mu^{(2)} \cdots \mu^{(d)}\, \gamma, v\Big) \qquad \text{for } m = 0,$$

where v is the concatenation of $v_1, v^{(1)}, \dots, v^{(d)}$, and possibly the empty sequence $v^{(d+1)}$ for $m = 0$. Therefore, $g(f((\mu, v))) = (\mu, v)$.

(ii) Now, we prove $f(g(\pi)) = \pi$ for every $\pi \in \mathcal{P}_\Lambda(n, -m)$, where $m \ge 0$ and $n \ge 1$. Let $(\mu, v) = g(\pi)$. For $n = 1$ and $m = 0$, we have $\pi = U_h V^h$, for certain up step U_h in $\Lambda_{\ge 0}$. Assume that $(h, (h))$ is the v_1th pair in the set $\mathcal{H}(0, 0, 0)$, which is $\Lambda_{\ge 0}$ in this case. By the definition of g, we see that $g(\pi) = (U_0, (v_1))$. On the other hand, the definition of f yields $f((\mu, v)) = U_h V^h$, which is π, as claimed. For $n = 1$ and $m = 1$, we have $\pi = S_h V^{h+1}$ for certain $S_h \in \Lambda_{\ge -1}$. Further, because S_h is the v_1th step in $\Lambda_{\ge -1}$, we have $g(\pi) = (D_1, (v_1))$, and $f(g(\pi)) = S_h V^{h+1}$, as claimed. If $m \ge 2$ and $D_m \in \Lambda$, then $\pi = D_m$ and $g(\pi) = (D_m, (1))$. It follows that $f(g(\pi)) = \pi$.

Herein, assume that $n \ge 2$ and suppose that the path π can be decomposed as in (2.25) for $m \ge 1$ or (2.27) for $m = 0$. The shape of μ is (m, d, k) in both cases. Let $c = (c_1, c_2, \dots, c_{d+1})$ and $k = h - c_1 - c_2 - \cdots - c_{d+1}$. Assume that (h, c) is the v_1th pair in $\mathcal{H}(m,d,k)$. By the definition of g, we have

$$g(\pi) = \Big(U_k\, \mu^{(1)}\, \mu^{(2)} \cdots \mu^{(d)}, v\Big) \qquad \text{for } m \geq 1,$$
$$g(\pi) = \Big(U_k\, \mu^{(1)}\, \mu^{(2)} \cdots \mu^{(d)}\gamma, v\Big) \qquad \text{for } m = 0,$$

where $(\mu^{(i)}, v^{(i)}) = g(\pi^{(i)})$ for $1 \leq i \leq d$ and possibly the empty path $(\gamma, v^{(d+1)}) = g(\gamma')$. Further, v is the concatenation of $v_1, v^{(1)}, \ldots, v^{(d)}$, and possibly the empty sequence $v^{(d+1)}$ for $m = 0$. On the other hand, if $(\mu, v) = g(\pi)$, then μ can be decomposed as

$$\mu = U_k\, \hat{\mu}^{(1)}\, \hat{\mu}^{(2)} \cdots \hat{\mu}^{(t)} \qquad \text{for } m \geq 1,$$
$$\mu = U_k\, \hat{\mu}^{(1)}\, \hat{\mu}^{(2)} \cdots \hat{\mu}^{(t)}\hat{\gamma} \qquad \text{for } m = 0,$$

for certain primary subpaths $\mu^{(i)} \in \mathcal{P}_\Gamma(n_i, -m_i)$ such that $m_i, n_i \geq 1$ for $i \in \{1, \ldots, t\}$, and possibly the empty 0-primary path $\hat{\gamma}$ for $m = 0$. We need to show that $d = t$, $\mu^{(i)} = \hat{\mu}^{(i)}$ for $1 \leq i \leq d$, and $\gamma = \hat{\gamma}$ if $m = 0$. For $1 \leq i \leq d$, the path $\mu^{(i)}$ and $\hat{\mu}^{(i)}$ are primary paths. Observe that if $m \geq 1$, then every m-primary path cannot be a prefix of any other m-primary path. For $1 \leq i \leq d$, we have $m_i \geq 1$, which implies that $\mu^{(1)} = \hat{\mu}^{(1)}$. It follows that $\mu^{(2)} = \hat{\mu}^{(2)}$, and so on up to $\mu^{(d)} = \hat{\mu}^{(d)}$, and thus $t = d$. Additionally, if $m = 0$, then $\gamma = \hat{\gamma}$.

By the induction hypothesis, we have $f(g(\pi^{(i)})) = \pi^{(i)}$ for $1 \leq i \leq d$, and $f(g(\gamma')) = \gamma'$ if $m = 0$. Under the assumption that (h, c) is the v_1th pair in $\mathcal{H}(m,d,k)$, we show that $f(\mu) = \pi$ and conclude that $f(g(\pi)) = \pi$. $\square$

2.4 Primary and free paths

Let a be a sequence of n integers $a_1, \ldots, a_n$ that sums to one. A partial sum of a is the sum $a_1 + \cdots + a_k$ for every $k \in \{1, \ldots, n\}$. Raney [31] showed that there is only one cyclic shift $a' = (a_k, a_{k+1}, \ldots, a_n, a_1, \ldots, a_{k-1})$ of a such that every partial sum of a' is positive (see also Graham et al. [21, p. 360]). Moreover, these cyclic shifts are all different. This lemma also appears in the literature as the cycle lemma [10]. For our purposes, we reformulate this lemma as follows.

Lemma 2.16 (Raney lemma [31]). *Let $b = (b_1, \ldots, b_n)$ be a sequence of integers whose sum is -1. There is only one cyclic shift b' of b such that every partial sum of b' except the total sum is nonnegative. Moreover, these shifts are all different.*

Proof. Observe that if we rearrange the terms of b in reverse order and negate them, then we obtain the sequence $(-b_n, -b_{n-1}, \ldots, -b_1)$, whose sum is $+1$, and from the

Raney lemma, we see that there is only one cyclic shift of such a modified sequence that has the property that every partial sum except the total sum is nonnegative. □

Recall that $\Lambda \subseteq \{V, S_N, S_{N-1}, \ldots\}$ such that $V, S_N \in \Lambda$ for fixed $N \geq 0$. The Raney lemma implies that

$$|\mathcal{P}_{\Lambda\setminus\{V\}}(n,-1)| = \frac{1}{n}|\mathcal{F}_{\Lambda\setminus\{V\}}(n,-1)|, \qquad (n \geq 1). \tag{2.29}$$

We extend this connection between 1-primary and free $(\Lambda \setminus \{V\})$-paths to the corresponding families of Λ-paths with vertical steps.

Theorem 2.17. *Let $\Lambda \subseteq \{V, S_N, S_{N-1}, \ldots\}$ such that $S_N, V \in \Lambda$ for fixed $N \geq 0$. For $n \geq 1$, we have*

$$|\mathcal{P}_\Lambda(n,-1)| = \frac{1}{n}\Big(|\mathcal{F}_\Lambda(n,-1)| - |\mathcal{F}_\Lambda(n,0)|\Big). \tag{2.30}$$

Proof. Any 1-primary path π in $\mathcal{P}_\Lambda(n,-1)$ has exactly n non-vertical steps and some number of vertical steps between them. Note that the first step of π is not V. This implies that π can be represented as $S_{a_1}V^{b_1}S_{a_2}V^{b_2}\cdots S_{a_n}V^{b_n}$ for some integers $a_1, \ldots, a_n$ in the set $\{k : S_k \in \Lambda\}$ and $b_1, \ldots, b_n \geq 0$. Let $\alpha = (a_1-b_1, a_2-b_2, \ldots, a_n-b_n)$. The total sum of members of α is -1, and every partial sum, except the total sum, is nonnegative.

Let $\beta = (\beta_1, \ldots, \beta_n)$ be a sequence of n subpaths $\beta_i = S_{c_i}V^{d_i}$ such that $S_{c_i} \in \Lambda$, $d_i \geq 0$, and $(c_1 - d_1) + (c_2 - d_2) + \cdots + (c_n - d_n) = -1$. Observe that β designates a free Λ-path running from $(0,0)$ to $(n,-1)$ in which the first step is non-vertical. Consider all n cyclic shifts of β. The Ranney lemma implies that every two of these n cyclic shifts are different and that there is exactly one cyclic shift of β that designates a 1-primary Λ-path. Moreover, any free Λ-path running from $(0,0)$ to $(n,-1)$ in which the first step is non-vertical can be represented by such a sequence β. Therefore, the number of 1-primary paths in $\mathcal{P}_\Lambda(n,-1)$ is equal to $1/n$ times the number of sequences β. The number of sequences β is $|\mathcal{F}_\Lambda(n,-1)| - |\mathcal{F}_\Lambda(n,0)|$, which gives the desired formula. □

Theorem 2.18. *Let $\Lambda \subseteq \{V, S_N, S_{N-1}, \ldots\}$ such that $S_N, V \in \Lambda$ for fixed $N \geq 0$. For $n \geq 1$ and $m \in \mathbb{Z}$, we have*

$$|\mathcal{F}_\Lambda(n,-m)| = \sum_{j=0}^{Nn+m} \binom{n+j}{j} |\mathcal{F}_{\Lambda\setminus\{V\}}(n, j-m)|, \tag{2.31a}$$

$$|\mathcal{P}_\Lambda(n,-1)| = \frac{1}{n}\sum_{j=0}^{Nn+1} \binom{n+j-1}{j} |\mathcal{F}_{\Lambda\setminus\{V\}}(n, j-1)|. \tag{2.31b}$$

Proof. First, we show (2.31a). The number of vertical steps in a path of $\mathcal{F}_\Lambda(n,-m)$ is an integer in $\{0, 1, \ldots, Nn+m\}$. Therefore, we partition the family $\mathcal{F}_\Lambda(n,-m)$ into pairwise disjoint subfamilies $A_0, A_1, \ldots, A_{Nn+m}$ such that A_j contains these in which the number of vertical steps is j. To calculate the size of A_j, observe that adding j vertical steps to any free $(\Lambda \setminus \{V\})$-path (without vertical steps) running from $(0,0)$ to $(n, j-m)$, we obtain a free path in $\mathcal{F}_\Lambda(n,-m)$. Any such path has n non-vertical steps and those j vertical steps may be added between them on s ways, where s is the number of solutions of $a_0 + a_1 + \cdots + a_n = j$, with $a_0, \ldots, a_n \geq 0$. Therefore, the size of A_j is $\binom{n+j}{j}$ times the size of $\mathcal{F}_{\Lambda\setminus\{V\}}(n, j-m)$.

The second equality (2.31b) follows directly from (2.30) together with (2.31a), i.e.,

$$\begin{aligned}
|\mathcal{P}_\Lambda(n,-1)| &= \frac{1}{n}\left(\sum_{j=0}^{Nn+1}\binom{n+j}{j}|\mathcal{F}_{\Lambda\setminus\{V\}}(n,j-1)| - \sum_{j=1}^{Nn+1}\binom{n+j-1}{j-1}|\mathcal{F}_{\Lambda\setminus\{V\}}(n,j-1)|\right)\\
&= \frac{1}{n}\left(|\mathcal{F}_{\Lambda\setminus\{V\}}(n,-1)| + \sum_{j=1}^{Nn+1}\left(\binom{n+j}{j} - \binom{n+j-1}{j-1}\right)|\mathcal{F}_{\Lambda\setminus\{V\}}(n,j-1)|\right)\\
&= \frac{1}{n}\left(|\mathcal{F}_{\Lambda\setminus\{V\}}(n,-1)| + \sum_{j=1}^{Nn+1}\binom{n+j-1}{j}|\mathcal{F}_{\Lambda\setminus\{V\}}(n,j-1)|\right),
\end{aligned}$$

and the formula follows. □

Theorem 2.19. *Let $\Sigma \subseteq \{V, S_N, S_{N-1}, \ldots\}$ for fixed $N \geq 0$, and $n \geq 1$. Recall that $\#Steps(S \in \mathcal{P}_\Sigma(n,-1))$ denotes the total number of occurrences of the step S in all paths of $\mathcal{P}_\Sigma(n,-1)$.*

(i) If $V \in \Sigma$, then $\#Steps(V \in \mathcal{P}_\Sigma(n,-1)) = |\mathcal{F}_\Sigma(n,0)|$.

(ii) If $S_k \in \Sigma$, then $\#Steps(S_k \in \mathcal{P}_\Sigma(n,-1)) = |\mathcal{F}_\Sigma(n-1,-k-1)|$.

Proof. Let S be a fixed step in Σ, and let us introduce the temporary notation

$$\mathcal{F} = \begin{cases} \mathcal{F}_\Sigma(n,0) & \text{if } S = V,\\ \mathcal{F}_\Sigma(n-1,-k-1) & \text{if } S = S_k \text{ for certain } k \in \mathbb{Z}.\end{cases}$$

We define the function ϕ from the set of all occurrences S in the paths of $\mathcal{P}_\Sigma(n,-1)$ to the set $\mathcal{F}$ as follows. Let $\pi \in \mathcal{P}_\Lambda(n,-1)$, and suppose that π has exactly d steps S and $d \geq 1$. For each $p \in \{1, \ldots, d\}$, the path π can be represented as

$$\pi = \underbrace{\pi^{(1)} S\, \pi^{(2)} S \cdots S\, \pi^{(p-1)} S\, \pi^{(p)}}_{\alpha}\, S\, \underbrace{\pi^{(p+1)} S \cdots \pi^{(d)} S\, \pi^{(d+1)}}_{\beta}, \tag{2.32}$$

for certain possibly empty subpaths $\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(d+1)}$ (see Figure 2.16). We set

$$\phi(\pi, p) = \beta\, \alpha, \tag{2.33}$$

where α and β are defined as in (2.32). To show that ϕ is a bijection, we need to show that $\phi(\pi, p)$ is a path in $\mathcal{F}$, and every path μ in $\mathcal{F}$ can be uniquely decomposed as $\mu = \beta\alpha$ in such a way that $\alpha S \beta \in \mathcal{P}_\Sigma(n, -1)$.

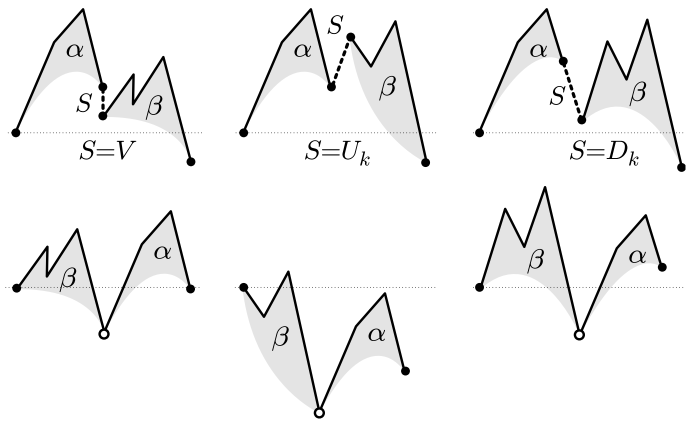

FIGURE 2.16: Three 1-primary Σ-paths (upper paths) and the results for the function ϕ from the proof of Theorem 2.19 (lower paths) for three cases: $S = V$ (left), $S = U_k$ (center), and $S = D_k$ (right). The minimal points (x, y) are denoted by open circles.

First, observe that $\phi(\pi, p)$ removes only one step S of π, which implies that the result is a free path in $\mathcal{F}$. Next, suppose that (x, y) is the leftmost point of $\phi(\pi, p)$ such that y is the minimal level (the horizontal line) that the path reaches. We shall prove that the path $\phi(\pi, p)$ reaches (x, y) exactly after the last step of β. Recall that π is a primary Σ-path running from $(0, 0)$ to $(n, -1)$ in which only the ending point lies below the x-axis. Thus, π reaches the lowest level exactly after part $\pi^{(d+1)}$ in (2.32). It follows that α is a path that does not go below the x-axis. On the other hand, only the ending point of β reaches the lowest level. It follows that $p - 1$ is the number of steps S of $\phi(\pi, p)$ that lie to the right of (x, y).

Let γ be a free Σ-path in $\mathcal{F}$ and $\gamma = \beta\alpha$ such that the last point of the subpath β lies at the leftmost minimal level reached by γ. We set $\phi^{-1}(\gamma)$ to be the pair $(\alpha S \beta, p)$, where p is the number of steps S in α plus one. □

Theorem 2.20. *Let* $\Sigma \subseteq \{V, S_N, S_{N-1}, \ldots\}$. *If* $n \geq 1$, *then* $\#Steps(\mathcal{P}_\Sigma(n, -1)) = |\mathcal{F}_\Sigma(n, -1)|$, *where* $\#Steps(\mathcal{P}_\Sigma(n, -1))$ *denotes the total number of all steps in the paths of* $\mathcal{P}_\Sigma(n, -1)$.

Proof. We show that there is a bijection ψ from the set of all steps in the paths of $\mathcal{P}_\Sigma(n, -1)$ to the set of paths in $\mathcal{F}_\Sigma(n, -1)$. Take a path μ in $\mathcal{P}_\Sigma(n, -1)$ and suppose that $\mu = \mu_1 \cdots \mu_r$. Letting $k \in \{1, 2, \ldots, r\}$, we set

$$\psi(\mu, k) = \mu_k \, \mu_{k+1} \cdots \mu_r \, \mu_1 \, \mu_2 \cdots \mu_{k-1}.$$

It is clear that $\psi(\mu, k) \in \mathcal{F}_\Sigma(n, -1)$. Next, we give a map ζ from $\mathcal{F}_\Sigma(n, -1)$ to the set of all steps in the paths of $\mathcal{P}_\Sigma(n, -1)$. Let $\pi \in \mathcal{F}_\Sigma(n, -1)$ and $\pi = \pi_1 \cdots \pi_r$. Let us represent π as the sequence $\hat{s} = (\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_r)$ of integers according to the rule

$$\hat{s}_i = \begin{cases} k & \text{if } \pi_i = S_k \text{ for certain } k \in \mathbb{Z}, \\ -1 & \text{if } \pi_i = V. \end{cases}$$

The sum of the sequence $\hat{s}$ is -1. Therefore, the modified Raney lemma (Lemma 2.16) implies that there is only one cyclic shift $s = (s_1, \ldots, s_r)$ of $\hat{s}$ such that each of its partial sums except the total sum is nonnegative. Moreover, this cyclic shift s uniquely determines an index k such that the cyclic shift $(s_k, s_{k+1}, \ldots, s_r, s_1, \ldots, s_{k-1})$ of s is the original sequence $\hat{s}$. Now, if we change the terms of the sequence s back into steps, we obtain a primary Σ-path μ. This implies that for every free path π in $\mathcal{F}_\Sigma(n, -1)$, we have uniquely associated a path μ in $\mathcal{P}_\Sigma(n, -1)$ and an index k such that $\phi(\mu, k) = \pi$. $\square$

2.5 Counting paths in a general case

Recall that $\Lambda \subseteq \{V, S_N, S_{N-1}, \ldots\}$ such that $V, S_N \in \Lambda$ for fixed $N \geq 0$. In this section, we consider the case where the set of steps Λ may contain infinitely many down steps. First, let us observe that for $n \geq 0, m \leq Nn$, and $(m, n) \neq (0, 0)$, the last step of every free path in $\mathcal{F}_\Lambda(n, m)$ is V or $S_k \in \Lambda$ for certain $k \in \mathbb{Z}$. The remaining steps of the path designate a free path in $\mathcal{F}_\Lambda(n, m+1)$ or $\mathcal{F}_\Lambda(n-1, m-k)$, respectively. This implies that the number of free Λ-paths running from $(0, 0)$ to (n, m) satisfies the following recurrence relation:

(i) we have $|\mathcal{F}_\Lambda(0, 0)| = 1$;

(ii) for all $n < 0$ or $m > Nn$, we have $|\mathcal{F}_\Lambda(n, m)| = 0$; and

(iii) for all $n \geq 0$ and $m \leq Nn$ such that $(n, m) \neq (0, 0)$, we have

$$|\mathcal{F}_\Lambda(n, m)| = |\mathcal{F}_\Lambda(n, m+1)| + \sum_{S_k \in \Lambda} |\mathcal{F}_\Lambda(n-1, m-k)|. \tag{2.34}$$

Observe that the second condition (ii) ensures that for all $n, m \geq 0$, even though Λ has infinitely many down steps, the sum on the right-hand side of (2.34) is finite.

Let us define a bivariate generating function

$$F_\Lambda(x, y) = \sum_{m \geq 0} \sum_{n \geq 0} |\mathcal{F}_\Lambda(n, Nn - m)| x^n y^m. \tag{2.35}$$

Following Wilf [41], we denote by $[x^n]f(x)$ the coefficient of x^n in the power series expansion of $f(x)$. Similarly, we denote by $[x^ny^m]f(x,y)$ the coefficient of x^ny^m in the power series expansion of the bivariate generating function $f(x,y)$. For instance, using this notation, we have

$$\begin{aligned}
[x^ny^m]F_\Lambda(x,y) &= |\mathcal{F}_\Lambda(n,Nn-m)|,\\
[x^ny^{Nn-m}]F_\Lambda(x,y) &= |\mathcal{F}_\Lambda(n,m)|,\\
[x^n]F_\Lambda(x,y) &= \sum_{m\geq 0}|\mathcal{F}_\Lambda(n,Nn-m)|y^m.
\end{aligned}$$

Proposition 2.21. *We have*

$$F_\Lambda(x,y) = \Big(1-y-x\sum_{S_k\in\Lambda}y^{N-k}\Big)^{-1}. \tag{2.36}$$

Proof. This directly follows from the recurrence relation (2.34). Namely, substituting $m-Nn$ for m in (2.34), we obtain

$$|\mathcal{F}_\Lambda(n,Nn-m)| = |\mathcal{F}_\Lambda(n,Nn-m+1)| + \sum_{S_k\in\Lambda}|\mathcal{F}_\Lambda(n-1,Nn-m-k)|.$$

Multiplying both sides by x^ny^m and summing over all $n,m\geq 0$ such that $(m,n)\neq(0,0)$, we obtain the following functional equation:

$$F_\Lambda(x,y)-1 = yF_\Lambda(x,y)+x\sum_{S_k\in\Lambda}y^{N-k}F_\Lambda(x,y).$$

Simplifying the formula, we obtain (2.36). □

Theorem 2.22. *For all $n\geq 1$ and $m\in\mathbb{Z}$, we have*

$$|\mathcal{F}_{\Lambda\setminus\{V\}}(n,m)| = [y^{Nn-m}]\Big(\sum_{S_k\in\Lambda}y^{N-k}\Big)^n, \tag{2.37a}$$

$$|\mathcal{F}_\Lambda(n,m)| = [y^{Nn-m}]\frac{1}{(1-y)^{n+1}}\Big(\sum_{S_k\in\Lambda}y^{N-k}\Big)^n, \tag{2.37b}$$

$$|\mathcal{P}_\Lambda(n,-1)| = \frac{1}{n}[y^{Nn+1}]\frac{1}{(1-y)^n}\Big(\sum_{S_k\in\Lambda}y^{N-k}\Big)^n. \tag{2.37c}$$

Proof. We first show (2.37a). Let μ be a free $(\Lambda\setminus\{V\})$-path running from $(0,0)$ to (n,m) and $\mu = S_{a_1}S_{a_2}\cdots S_{a_n}$, where each $S_{a_i}\in\Lambda$ and $a_1+\cdots+a_n=m$. Recall that N is the maximal integer h such that $S_h\in\Lambda$. It follows that μ can be represented as $S_{N-b_1}S_{N-b_2}\cdots S_{N-b_n}$, where $b_i=N-a_i$, and b_i is a nonnegative integer in $\{N-k : S_k\in\Lambda\}$. Moreover, the sum $b_1+\cdots+b_n$ is $Nn-m$.

On the other hand, every sequence $(c_1, c_2, \dots, c_n)$ of n nonnegative integers such that $c_i \in \{N-k : S_k \in \Lambda\}$ and whose sum is $Nn-m$ uniquely determines a free $(\Lambda \setminus \{V\})$-path $S_{N-c_1}S_{N-c_2}\cdots S_{N-c_n}$ in $\mathcal{F}_{\Lambda\setminus\{V\}}(n,m)$. This correspondence is a one-to-one correspondence; therefore, instead of directly counting free paths in $\mathcal{F}_{\Lambda\setminus\{V\}}(n,m)$, we derive the number of sequences of n nonnegative integers over $\{N-k : S_k \in \Lambda\}$, whose sum is $Nn-m$. This number is equal to the coefficient of y^{Nn-m} in the power series expansion of $(\sum_{S_k\in\Lambda} y^{N-k})^n$, as claimed.

To prove (2.37b), we use the bivariate generating function (2.36) to obtain

$$\sum_{m\geq 0} |\mathcal{F}_\Lambda(n, Nn-m)| y^m = [x^n]F_\Lambda(x,y) = \frac{1}{(1-y)^{n+1}}\Big(\sum_{S_k\in\Lambda} y^{N-k}\Big)^n, \tag{2.38}$$

and (2.37b) follows. To show (2.37c), we apply (2.30) to obtain $|\mathcal{P}_\Lambda(n,-1)| = (|\mathcal{F}_\Lambda(n,-1)| - |\mathcal{F}_\Lambda(n,0)|)/n$. Substituting (2.37b) and simplifying, we obtain the formula. □

Corollary 2.23. *Let $n \geq 1$. The expected number of vertical steps in a path of $\mathcal{P}_\Lambda(n,-1)$ is equal to*

$$n \cdot \frac{|\mathcal{F}_\Lambda(n,0)|}{|\mathcal{F}_\Lambda(n,-1)| - |\mathcal{F}_\Lambda(n,0)|}. \tag{2.39}$$

The expected number of all steps in a path of $\mathcal{P}_\Lambda(n,-1)$ is equal to

$$n \cdot \frac{|\mathcal{F}_\Lambda(n,-1)|}{|\mathcal{F}_\Lambda(n,-1)| - |\mathcal{F}_\Lambda(n,0)|}. \tag{2.40}$$

Proof. The expected number of vertical steps in a path of $\mathcal{P}_\Lambda(n,-1)$ is equal to the number of all vertical steps in all paths of $\mathcal{P}_\Lambda(n,-1)$ divided by the number of paths in $\mathcal{P}_\Lambda(n,-1)$. By Theorem 2.19, this number is equal to $|\mathcal{F}_\Lambda(n,0)|/|\mathcal{P}_\Lambda(n,-1)|$. Applying (2.30) we obtain the first formula. On the other hand, the expected number of all steps in a path of $\mathcal{P}_\Lambda(n,-1)$ is equal to the number of steps in all paths of $\mathcal{P}_\Lambda(n,-1)$ divided by the number of paths in $\mathcal{P}_\Lambda(n,-1)$. By Theorem 2.20, this number is equal to $|\mathcal{F}_\Lambda(n,-1)|/|\mathcal{P}_\Lambda(n,-1)|$. Applying (2.30) we obtain the second formula. □

2.6 Counting paths with a finite set of steps

In this section, we consider the case wherein the set of steps Λ is finite. Namely, throughout this section, we assume that $\Lambda \subseteq \{V, S_N, S_{N-1}, \dots, S_{-K}\}$ such that $S_N, S_{-K}, V \in \Lambda$ for fixed $N, K \geq 0$.

First, let us observe that for all $m > \max\{1, K\}$ and $n \geq 0$, the set $\mathcal{P}_\Lambda(n,-m)$ is empty. In addition, note that if $K = 0$, then the last step of every path in $\mathcal{P}_\Lambda(n,-1)$, with $n \geq 1$, is the vertical step V. It is worth noting that we study primary Λ-paths under the assumption that $\mathcal{P}_\Lambda(0,0) = \{\lambda\}$, where λ is the empty path, and $\mathcal{P}_\Lambda(0,-m) = \emptyset$ for $m \geq 1$. Thus, $|\mathcal{P}_\Lambda(0,0)| = 1$ and $|\mathcal{P}_\Lambda(0,-m)| = 0$ for $m \geq 1$.

Proposition 2.24. *Let $n \geq 1$. If $K = 0$, then $|\mathcal{P}_\Lambda(n,0)| = |\mathcal{P}_\Lambda(n,-1)|$. If $K = 1$, then*

$$|\mathcal{P}_\Lambda(n,0)| = (-1)^n + \sum_{j=1}^{n}(-1)^{n-j}|\mathcal{P}_\Lambda(j,-1)|. \tag{2.41}$$

Proof. If $K = 0$, then Λ does not contain any down step. Thus, the last step of any path in $\mathcal{P}_\Lambda(n,-1)$ is V. It follows that the remaining steps of the path form a path in $\mathcal{P}_\Lambda(n,0)$ and the formula follows.

Similarly, if $K = 1$, then Λ contains only one down step, i.e., D_1. Thus, the last step of any path in $\mathcal{P}_\Lambda(n,-1)$ is D_1 or V. It follows that $|\mathcal{P}_\Lambda(n,-1)| = |\mathcal{P}_\Lambda(n-1,0)|+|\mathcal{P}_\Lambda(n,0)|$ for $n \geq 1$, $|\mathcal{P}_\Lambda(0,-1)| = 0$, and $|\mathcal{P}_\Lambda(0,0)| = 1$. Moving the term $|\mathcal{P}_\Lambda(n-1,0)|$ to the left-hand side, we obtain a recurrence relation for $|\mathcal{P}_\Lambda(n,0)|$. Iterating this recurrence we derive the required sum. □

Proposition 2.25. *For $n \geq 1$, if $K \geq 2$ and $K = m$, then $|\mathcal{P}_\Lambda(n,-K)| = |\mathcal{P}_\Lambda(n-1,0)|$.*

Proof. This follows from the observation that the last step of any m-primary Λ-path running from $(0,0)$ to $(n,-m)$, where $m = K$ and $K \geq 2$, is D_K. Removing this step we obtain a 0-primary Λ-path running from $(0,0)$ to $(n-1,0)$. □

Definition 2.26. For $m \geq 0$, let

$$P_{\Lambda,m}(x) = \sum_{n\geq 0} |\mathcal{P}_\Lambda(n,-m)|x^n. \tag{2.42}$$

Subsequently, we obtain the functional equation for the generating function $P_{\Lambda,m}(x)$. However, first, let us introduce the following necessary notation:

$$\delta_{\Lambda,m} = \begin{cases} |\Lambda_{\geq -m}| & \text{if } m \in \{0,1\}, \\ |\Lambda \cap \{D_m\}| & \text{if } m \geq 2. \end{cases} \tag{2.43}$$

Theorem 2.27. *Let $\Lambda \subseteq \{V, S_N, S_{N-1}, \ldots, S_{-K}\}$ such that $U_N, D_K, V \in \Lambda$ for fixed $N, K \geq 0$. For $1 \leq m \leq \max\{1, K\}$, we have*

$$\begin{aligned} P_{\Lambda,0}(x) &= 1 + \delta_{\Lambda,0}xP_{\Lambda,0}(x) + xP_{\Lambda,0}(x)\sum_{k=1}^{N}\sum_{d=1}^{k}|\mathcal{H}_\Lambda(0,d,k)|\sum_{M}\prod_{j=1}^{d}P_{\Lambda,m_j}(x), \\ P_{\Lambda,m}(x) &= \delta_{\Lambda,m}x + x\sum_{k=0}^{N}\sum_{d=1}^{k+1}|\mathcal{H}_\Lambda(m,d,k)|\sum_{M}\prod_{j=1}^{d}P_{\Lambda,m_j}(x), \end{aligned} \tag{2.44}$$

where the summation range M is over all solutions of

$$1 \le m_1, \dots, m_{d-1} \le K, \quad m \le m_d \le K, \quad m_1 + \cdots + m_d = k + m. \tag{2.45}$$

Proof. By Theorem 2.10, the number of m-primary Λ-paths in $\mathcal{P}_\Lambda(n, -m)$ is equal to the number of weighted m-primary Γ-paths in $\mathcal{W}_\Gamma^\Lambda(n, -m)$, where $\Gamma = (\Lambda \setminus \{V\}) \cup \{D_1, U_0, \dots, U_N\}$. It follows that

$$P_{\Lambda,m}(x) = \sum_{n \ge 0} |\mathcal{W}_\Gamma^\Lambda(n, -m)| x^n.$$

Every path in $\mathcal{W}_\Gamma^\Lambda(n, -m)$ consists of (only non-vertical) steps from Γ, and the decomposition of such paths directly translates to the functional equation for the generating function that counts these paths according to the length.

First, let $m = 0$ and $(\mu, v) \in \mathcal{W}_\Gamma^\Lambda(n, -m)$. If $n = 0$, then $\mu = \lambda$, and thus, the constant term of $P_{\Lambda,0}(x)$ is one. If $n = 1$, then $\mu = U_0$ and $v = (v_1)$ such that $1 \le v_1 \le |\Lambda_{\ge 0}|$. For $n \ge 2$, suppose that μ is decomposable as

$$\mu = U_k \, \mu^{(1)} \cdots \mu^{(d)} \, \gamma,$$

for certain $0 \le d \le k \le N$ and that (i) for $1 \le i \le d$, we have $\mu^{(i)} \in \mathcal{P}_\Gamma(n_i, -m_i)$ for certain $m_i, n_i \ge 1$, (ii) $\gamma \in \mathcal{P}_\Gamma(n_{d+1}, 0)$ for certain $n_{d+1} \ge 0$, and (iii) $m_1 + \cdots + m_d = k + m$ and $1 + n_1 + \cdots + n_{d+1} = n$ (see (2.4) in Section 2.1). Recall that the weight of μ is the product of the maximal weights of steps $\mu_1, \dots, \mu_n$ and that the maximal weight of the first step $\mu_1 = U_k$ is $|\mathcal{H}_\Lambda(m, d, k)|$.

For $m, d, k \ge 0$ and $1 \le m_1, \dots, m_d \le K$ such that $m_1 + \cdots + m_d = k + m$ and $m_d \ge m$, let us denote by $A_{m,d,k}^{m_1,\dots,m_d}(n)$ the number of weighted paths (μ, v) in $\mathcal{W}_\Gamma^\Lambda(n, -m)$, where μ has the decomposition given by (2.4) for $m = 0$ or given by (2.3) for $m \ge 1$. Observe that if $k = 0$ and $n \ge 2$, then $\mu = U_0\gamma$, and thus, $d = 0$ and $\gamma \in \mathcal{P}_\Gamma(n_{d+1}, 0)$, with $n_{d+1} \ge 1$. It follows that

$$A_{0,0,0}(n) = [x^n] |\mathcal{H}_\Lambda(0,0,0)| x (P_{\Lambda,0}(x) - 1) = [x^n] |\Lambda_{\ge 0}| x (P_{\Lambda,0}(x) - 1).$$

For $k \ge 1$ and $d \ge 1$, we have

$$A_{0,d,k}^{m_1,\dots,m_d}(n) = [x^n] \big(|\mathcal{H}_\Lambda(0,d,k)| x P_{\Lambda,m_1}(x) P_{\Lambda,m_2}(x) \cdots P_{\Lambda,m_d}(x) \big) P_{\Lambda,0}(x).$$

Summing over all possible shapes (m, d, k) and $m_1, \ldots, m_d$ satisfying (2.45), we obtain

$$\begin{aligned} P_{\Lambda,0}(x) &= 1 + |\Lambda_{\geq 0}|x + |\Lambda_{\geq 0}|x(P_{\Lambda,0}(x) - 1) + \\ &+ x\left(\sum_{k=1}^{N}\sum_{d=1}^{k}|\mathcal{H}_\Lambda(m,d,k)|\sum_{M} P_{\Lambda,m_1}(x)P_{\Lambda,m_2}(x)\cdots P_{\Lambda,m_d}(x)\right)P_{\Lambda,0}(x). \end{aligned}$$

Simplifying, we obtain the required functional equation for the case $m = 0$.

Similarly, we show the functional equation for $P_{\Lambda,m}(x)$ for $m \geq 1$. We only note that in this case the constant term of $P_{\Lambda,m}(x)$ is zero, the number of paths in $\mathcal{W}_\Gamma^\Lambda(1, -m)$ is $\delta_{\Lambda,m}$, and for $n \geq 2$, according to the decomposition of an m-primary Γ-path, with $m \geq 1$ (see (2.3) in Section 2.1), we have $1 \leq d \leq k + 1$ and

$$A_{m,d,k}^{m_1,\ldots,m_d}(n) = [x^n]|\mathcal{H}_\Lambda(m,d,k)|xP_{\Lambda,m_1}(x)P_{\Lambda,m_2}(x)\cdots P_{\Lambda,m_d}(x).$$

□

Corollary 2.28. *If $K \in \{0, 1\}$, then*

$$\begin{aligned} P_{\Lambda,0}(x) &= 1 + |\Lambda_{\geq 0}|xP_{\Lambda,0}(x) + xP_{\Lambda,0}(x)\sum_{k=1}^{N}\sum_{U_h\in\Lambda_{\geq k}}\binom{h}{k}P_{\Lambda,1}(x)^k, \\ P_{\Lambda,1}(x) &= |\Lambda_{\geq -1}|x + x\sum_{k=0}^{N}\sum_{U_h\in\Lambda_{\geq k}}\binom{h+1}{k+1}P_{\Lambda,1}(x)^{k+1}, \end{aligned} \tag{2.46}$$

Proof. First, we apply Theorem 2.27. Further simplifications follow from the assumption that $K \in \{0, 1\}$. In this case, observe that any 0-primary Γ-path in which the first step is U_k decomposes into exactly k nonempty 1-primary Γ-paths and possibly the empty 0-primary Γ-path. Similarly, any 1-primary path in which the first step is U_k decomposes into exactly $k + 1$ nonempty 1-primary Γ-paths. □

2.7 Łukasiewicz paths

In this section, we consider $\Lambda_1 = \{V, U_N, U_{N-1}, \ldots, U_0, D_1\}$ for fixed $N \geq 0$. According to (2.5), we set $\Gamma_1 = \Lambda_1 \setminus \{V\}$. Recall that Γ_1-paths are N-Łukasiewicz paths. Łukasiewicz paths are the well-known families of lattice paths [19, 33, 37, 38]. It is worth noting that 1-Łukasiewicz paths are Motzkin paths [3, 11, 12].

First, observe that because D_1 is the only down step in Λ_1 and Γ_1, for all $m \geq 2$ and $n \geq 0$, the families $\mathcal{W}_{\Gamma_1}^{\Lambda_1}(n, -m)$ and $\mathcal{P}_{\Lambda_1}(n, -m)$ are empty. Let us consider the family

$\mathcal{W}_{\Gamma_1}^{\Lambda_1}(n,-m)$ of weighted N-Łukasiewicz paths for $m \in \{0,1\}$. Let $(\mu, v) \in \mathcal{W}_{\Gamma_1}^{\Lambda_1}(n,-m)$, where $\mu = \mu_1 \cdots \mu_n$ and $v = (v_1, \ldots, v_n)$.

- For $n = 0$, we have $\mathcal{W}_{\Gamma_1}^{\Lambda_1}(0,0) = \{\lambda\}$ and $\mathcal{W}_{\Gamma_1}^{\Lambda_1}(0,-1) = \emptyset$.
- For $n = 1$, we have $\mathcal{W}_{\Gamma_1}^{\Lambda_1}(1,-m) = \{(S_{-m}, (v_1)) : 1 \leq v_1 \leq N+1+m\}$.
- For $n \geq 2$ and $m = 0$, if the first step of μ is U_k, then $\mu = U_k\, \mu^{(1)} \cdots \mu^{(k)}\, \gamma$, where each $\mu^{(i)}$ is a 1-primary Γ_1-path, and γ is a 0-primary Γ_1-path. The weight v_1 of the first step μ_1 satisfies
$$1 \leq v_1 \leq w_\mu(1) = |\mathcal{H}_{\Lambda_1}(0,k,k)| = \binom{N+1}{k+1}.$$
- For $n \geq 2$ and $m = 1$, if the first step of μ is U_k, then $\mu = U_k\, \mu^{(1)} \cdots \mu^{(k+1)}$, where each $\mu^{(i)}$ is a 1-primary Γ_1-path. The weight v_1 of the first step μ_1 satisfies
$$1 \leq v_1 \leq w_\mu(1) = |\mathcal{H}_{\Lambda_1}(1,k+1,k)| = \binom{N+2}{k+2}.$$

Recall that the weight of the entire path μ is $w(\mu) = w_\mu(1) \cdots w_\mu(n)$. The weight function $w_\mu(i)$ (see Definition 2.4) in this case is defined as follows. For each $i \in \{1, \ldots, n\}$, we have

$$w_\mu(i) = \begin{cases} N+2, & \text{if } \mu_i = D_1, \\ \binom{N+2}{k+2} & \text{if } \mu_i = U_k \text{ and } (m = 1 \text{ or } \mu_i \text{ starts above the } x-\text{axis}), \\ \binom{N+1}{k+1} & \text{if } \mu_i = U_k \text{ and } (m = 0 \text{ and } \mu_i \text{ starts at the } x-\text{axis}). \end{cases} \tag{2.47}$$

Theorem 2.29. *For all $m \in \{0,1\}$ and $n \geq 0$, the number of weighted N-Łukasiewicz paths in $\mathcal{W}_{\Gamma_1}^{\Lambda_1}(n,-m)$ is equal to the number of non-weighted paths in $\mathcal{P}_{\Lambda_1}(n,-m)$. Specifically,*
$$|\mathcal{P}_{\Lambda_1}(n,-m)| = \sum_{\mu \in \mathcal{P}_{\Gamma_1}(n,-m)} w(\mu) = |\mathcal{W}_{\Gamma_1}^{\Lambda_1}(n,-m)|.$$

Proof. By Theorem 2.10, we see that for $\Lambda_1 = \{V, U_N, \ldots, U_0, D_1\}$ and $\Gamma_1 = \Lambda_1 \setminus \{V\}$, there is a bijection between $\mathcal{P}_{\Lambda_1}(n,-m)$ and $\mathcal{W}_{\Gamma_1}^{\Lambda_1}(n,-m)$. □

Remark. It is worth noting that the weight function over the steps in Γ_1 in a 0-primary Γ_1-path depends on the step U_k, N, and whether the path starts on the x-axis. For a 1-primary Γ_1-path, the weight function only depends on the step U_k and N.

Theorem 2.30. *For $m \in \{0, 1\}$, let $P_m(x) = \sum_{n\geq 0} |\mathcal{P}_{\Lambda_1}(n, -m)|x^n$. We have*

$$P_0(x) = 1 + xP_0(x) \sum_{k=0}^{N} (1 + P_1(x))^k, \tag{2.48a}$$

$$P_1(x) = x \sum_{k=0}^{N+1} (1 + P_1(x))^k. \tag{2.48b}$$

Proof. We apply Theorem 2.27 for the set of steps $\Lambda_1 = \{V, U_N, U_{N-1}, \ldots, U_0, D_1\}$. If $m = 0$, then the shape of each path $\mu \in \mathcal{P}_{\Gamma_1}(n, 0)$ is $(0, k, k)$. If $m = 1$, then the shape of each path $\mu \in \mathcal{P}_{\Gamma_1}(n, -1)$ is $(1, k+1, k)$. Further, each $\mu^{(i)}$ in the decomposition of μ is a 1-primary Γ_1-path, and thus, there is only one solution of (2.45). Therefore, the functional equation (2.44) simplifies as follows:

$$\begin{aligned}
P_0(x) &= 1 + (N+1)xP_0(x) + xP_0(x) \sum_{k=1}^{N} |\mathcal{H}_{\Lambda_1}(0, k, k)|(P_1(x))^k \\
&= 1 + (N+1)xP_0(x) + xP_0(x) \sum_{k=1}^{N} \binom{N+1}{k+1} (P_1(x))^k \\
&= 1 + xP_0(x) \sum_{k=0}^{N} \binom{N+1}{k+1} (P_1(x))^k \\
&= 1 + x\frac{P_0(x)}{P_1(x)} \sum_{k=1}^{N+1} \binom{N+1}{k} (P_1(x))^k \\
&= 1 + x\frac{P_0(x)}{P_1(x)} \left((1 + P_1(x))^{N+1} - 1\right)
\end{aligned}$$

and the formula follows. The formula for $P_1(x)$ can be proved in much the same way. $\square$

Corollary 2.31. *If $N = 1$, then*

$$P_0(x) = \frac{1 - x - \sqrt{1 - 6x - 3x^2}}{2x(1 + x)}, \quad P_1(x) = \frac{1 - 3x - \sqrt{1 - 6x - 3x^2}}{2x}.$$

Proof. Applying Theorem 2.30 for $N = 1$, we obtain $0 = 3x + (3x - 1)P_1(x) + xP_1(x)^2$. There are two solutions of this functional equation, $P_1^{\pm}(x) = (1 - 3x \pm \sqrt{1 - 6x - 3x^2})/2x$. According to the initial value $|\mathcal{P}_{\Lambda_1}(0, -1)| = 0$, the correct one is $P_1^{-}(x)$. On the other hand, by Theorem 2.30 for $m = 0$, we obtain $P_0(x) = 1/(1 - 2x - xP_1(x))$. $\square$

Proposition 2.32. *[14, Eq. 30a] If $N = 1$, then*

$$\sum_{n\geq 0} |\mathcal{F}_{\Lambda_1}(n, 0)|x^n = \frac{1}{\sqrt{1 - 6x - 3x^2}}, \quad \sum_{n\geq 0} |\mathcal{F}_{\Lambda_1}(n, -1)|x^n = \frac{1 - x - \sqrt{1 - 6x - 3x^2}}{2x\sqrt{1 - 6x - 3x^2}}.$$

Proposition 2.33. *For all $A, B, C \geq 0$, we have*

$$\frac{(1 \pm x^A)^B}{(1-x)^C} = \sum_{n \geq 0} \sum_{k=0}^{\lfloor n/A \rfloor} (\pm 1)^k \binom{B}{k} \binom{C+n-kA-1}{C-1} x^n. \tag{2.49}$$

Proof. Let us first consider the generating function $(1-x^A)^B/(1-x)^C$. Because this is the product of two generating functions, we start with the Cauchy product, and

$$\begin{aligned}
\frac{(1-x^A)^B}{(1-x)^C} &= \sum_{i=0}^{B} (-1)^i \binom{B}{i} x^{Ai} \sum_{j \geq 0} \binom{C+j-1}{j} x^j \\
&= \sum_{i=0}^{AB} \delta_{A,i} (-1)^{i/A} \binom{B}{i/A} x^i \sum_{j \geq 0} \binom{C+j-1}{C-1} x^j \\
&= \sum_{n \geq 0} \sum_{k=0}^{n} \delta_{A,k} (-1)^{k/A} \binom{B}{k/A} \binom{C+n-k-1}{C-1} x^n \\
&= \sum_{n \geq 0} \sum_{k=0}^{\lfloor n/A \rfloor} \delta_{A,kA} (-1)^k \binom{B}{k} \binom{C+n-kA-1}{C-1} x^n,
\end{aligned}$$

where $\delta_{i,j} = 1$ if $i|j$ and $\delta_{i,j} = 0$ if $i \not| j$. In the same manner we can obtain the formula for the plus sign. □

Theorem 2.34. *Let $\Lambda_1 = \{V, U_N, U_{N-1}, \ldots, U_0, D_1\}$. For all $m \in \mathbb{Z}$ and $n \geq 1$, we have*

$$|\mathcal{F}_{\Lambda_1}(n,m)| = \sum_{k=0}^{\lfloor \frac{Nn-m}{N+2} \rfloor} (-1)^k \binom{n}{k} \binom{(N+2)(n-k)-m}{2n}, \tag{2.50a}$$

$$|\mathcal{P}_{\Lambda_1}(n,-1)| = \frac{1}{n} \sum_{k=0}^{\lfloor \frac{Nn+1}{N+2} \rfloor} (-1)^k \binom{n}{k} \binom{(N+2)(n-k)}{2n-1}, \tag{2.50b}$$

$$|\mathcal{P}_{\Lambda_1}(n,0)| = (-1)^n + \sum_{j=1}^{n} \sum_{k=0}^{\lfloor \frac{Nj+1}{N+2} \rfloor} \frac{(-1)^{k+n-j}}{j} \binom{j}{k} \binom{(N+2)(j-k)}{2j-1}. \tag{2.50c}$$

Proof. Applying (2.37b) for the set $\Lambda_1 = \{V, U_N, U_{N-1}, \ldots, U_0, D_1\}$, we obtain

$$\begin{aligned}
|\mathcal{F}_{\Lambda_1}(n,m)| &= [x^{Nn-m}] \frac{1}{(1-x)^{n+1}} \Big(\sum_{k=-1}^{N} x^{N-k} \Big)^n \\
&= [x^{Nn-m}] \frac{(1+x+x^2+\cdots+x^{N+1})^n}{(1-x)^{n+1}} \\
&= [x^{Nn-m}] \frac{(1-x^{N+2})^n}{(1-x)^{2n+1}}.
\end{aligned}$$

Using Proposition 2.33, we obtain (2.50a). Similarly, we apply (2.37c) and Proposition 2.33 to get (2.50b). Having the formula for the number of paths in $\mathcal{P}_{\Lambda_1}(n,-1)$, we use Proposition 2.24 to get (2.50c). $\square$

Corollary 2.35. *The number of vertical steps in the set of paths $\mathcal{P}_{\Lambda_1}(n,-1)$ is equal to*

$$\sum_{k=0}^{\lfloor \frac{Nn}{N+2} \rfloor} (-1)^k \binom{n}{k} \binom{(N+2)(n-k)}{2n}.$$

The number of all steps in the set of paths $\mathcal{P}_{\Lambda_1}(n,-1)$ is equal to

$$\sum_{k=0}^{\lfloor \frac{Nn+1}{N+2} \rfloor} (-1)^k \binom{n}{k} \binom{(N+2)(n-k)+1}{2n}.$$

Proof. The first formula follows from Theorem 2.19 and Theorem 2.34. The second one from Theorem 2.20 and Theorem 2.34. $\square$

Example. If $N=1$, then

$$\begin{aligned}
(|\mathcal{F}_{\Lambda_1}(n,0)|)_{n\geq 0} &= (1,3,15,81,459,2673,15849,95175,576963,\ldots) && (A122868)\\
(|\mathcal{F}_{\Lambda_1}(n,-1)|)_{n\geq 0} &= (1,6,33,189,1107,6588,39663,240894,\ldots) && (A260774)\\
(|\mathcal{P}_{\Lambda_1}(n,0)|)_{n\geq 0} &= (1,2,7,29,133,650,3319,17498,94525,\ldots) && (A064641)\\
(|\mathcal{P}_{\Lambda_1}(n,-1)|)_{n\geq 0} &= (0,3,9,36,162,783,3969,20817,112023,\ldots) && (A156016)
\end{aligned}$$

The numbers starting with A denote corresponding sequences in OEIS [32].

2.8 Raney paths with vertical steps

In this section, we consider the case where the set of steps Λ_2 contains infinitely many down steps. Namely, we set $\Lambda_2 = \{V, S_N, S_{N-1}, \ldots\}$ for fixed $N \geq 0$. According to (2.5), we set $\Gamma_2 = \{S_N, S_{N-1}, \ldots\}$. Recall that Γ_2-paths are N-Raney paths considered in Chapter 3. It turns out that even though the set of steps Λ_2 contains infinitely many down steps, several formulas that count these paths have a simple form.

For $m \geq 0$ and $n \geq 0$, let us consider the family $\mathcal{W}_{\Gamma_2}^{\Lambda_2}(n,-m)$ of weighted N-Raney paths. If $n=0$, then $\mathcal{W}_{\Gamma_2}^{\Lambda_2}(0,0) = \{\lambda\}$ and $\mathcal{W}_{\Gamma_2}^{\Lambda_2}(0,-m) = \emptyset$ for $m \geq 1$. For $n \geq 1$, let $(\mu, v) \in W_{\Gamma_2}^{\Lambda_2}(m,n)$, where $\mu = \mu_1 \cdots \mu_n$. Recall that the weight of the path μ is $w(\mu) = w_\mu(1) \cdots w_\mu(n)$. By Corollary 2.7, for each $i \in \{1, \ldots, n\}$, we have

(i) if $\mu_i = D_p$, then

$$w_\mu(i) = \begin{cases} N+2 & \text{if } p = 1, \\ 1 & \text{if } p \geq 2, \end{cases}$$

(ii) if $\mu_i = U_k$ is the first step of the uniquely determined subpath σ of μ in which μ_i is the first step (see Lemma 2.2 (i) and (ii)), and if the shape of σ is (p, d, k), then

$$w_\mu(i) = \binom{N-k+d+1-\epsilon_p}{N-k},$$

where $\epsilon_p = 0$ if $p \in \{0, 1\}$ and $\epsilon_p = 1$ if $p \geq 2$.

Theorem 2.36. *For all $m \geq 0$ and $n \geq 0$, the number of weighted N-Raney paths in $\mathcal{W}_{\Gamma_2}^{\Lambda_2}(n, -m)$ is equal to the number of non-weighted paths in $\mathcal{P}_{\Lambda_2}(n, -m)$. Specifically,*

$$|\mathcal{P}_{\Lambda_2}(n, -m)| = \sum_{\mu \in \mathcal{P}_{\Gamma_2}(n,-m)} w(\mu) = |\mathcal{W}_{\Gamma_2}^{\Lambda_2}(n, -m)|.$$

Proof. This follows from Theorem 2.10 for Λ_2 and Γ_2. □

Theorem 2.37. *Let $\Lambda_2 = \{V, S_N, S_{N-1}, \ldots\}$. For all $m \in \mathbb{Z}$ and $n \geq 1$, we have*

$$|\mathcal{F}_{\Lambda_2}(n, m)| = \binom{(N+2)n - m}{2n}, \tag{2.51a}$$

$$|\mathcal{P}_{\Lambda_2}(n, -1)| = \frac{1}{n}\binom{(N+2)n}{2n-1}. \tag{2.51b}$$

Proof. Applying Theorem 2.22 for $\Lambda_2 = \{V, S_N, S_{N-1}, \ldots\}$, we obtain

$$|\mathcal{F}_{\Lambda_2}(n, m)| = [x^{Nn-m}]\frac{1}{(1-x)^{n+1}}(1 + x + x^2 + \cdots)^n = [x^{Nn-m}]\frac{1}{(1-x)^{2n+1}}.$$

Using the binomial expansion, we derive the number of free Λ_2-paths. Similarly, applying Theorem 2.22, we obtain the number of 1-primary Λ_2-paths. □

Corollary 2.38. *The expected number of vertical steps in a path of $\mathcal{P}_{\Lambda_2}(n, -1)$ is equal to $(Nn+1)/2$. The expected number of all steps in a path in $\mathcal{P}_{\Lambda_2}(n, -1)$ is equal to $((N+2)n+1)/2$.*

Proof. This follows from Corollary 2.23 and Theorem 2.37. □

Example. If $N = 1$, then

$$(|\mathcal{F}_{\Lambda_2}(n, 0)|)_{n\geq 0} = (1, 3, 15, 84, 495, 3003, 18564, 116280, \ldots) \qquad (A005809)$$

$$(|\mathcal{P}_{\Lambda_2}(n, -1)|)_{n\geq 0} = (0, 3, 10, 42, 198, 1001, 5304, 29070, 163438, \ldots) \qquad (A007226)$$

2.9 Dyck paths with vertical steps

In this section, we consider the set of steps $\Lambda_3 = \{V, U_N, D_K\}$ for fixed $N, K \geq 1$. According to (2.5), we set $\Gamma_3 = \{U_N, U_{N-1}, \dots, U_0, D_1, D_K\}$. Recall that if $N = K = 1$, then Γ_3-paths are Motzkin paths and $(\Lambda_3 \setminus \{V\})$-paths are Dyck paths.

For $m \geq 0$ and $n \geq 0$, let us consider the family $\mathcal{W}_{\Gamma_3}^{\Lambda_3}(n, -m)$ of weighted Γ_3-paths. For $n = 0$, we have $\mathcal{W}_{\Gamma_3}^{\Lambda_3}(0, 0) = \{\lambda\}$ and $\mathcal{W}_{\Gamma_3}^{\Lambda_3}(0, -m) = \emptyset$. For $n \geq 1$, let $(\mu, v) \in \mathcal{W}_{\Gamma_3}^{\Lambda_3}(n, -m)$, where $\mu = \mu_1 \cdots \mu_n$. Recall that the weight of μ is $w(\mu) = w_\mu(1) \cdots w_\mu(n)$. By Definition 2.4, for each $i \in \{1, \dots, n\}$, we have

(i) if $\mu_i = D_p$, then

$$w_\mu(i) = \begin{cases} 2 & \text{if } p = 1 \text{ and } K = 1, \\ 1 & \text{if } p = 1 \text{ and } K > 1, \\ 1 & \text{if } p \geq 2. \end{cases}$$

(ii) if $\mu_i = U_k$ is the first step of the uniquely determined subpath σ of μ in which μ_i is the first step (see Lemma 2.2 (i) and (ii)), and the shape of σ is (p, d, k), then

$$w_\mu(i) = \binom{N - k + d - \epsilon_p}{N - k}, \tag{2.52}$$

where $\epsilon_p = 0$ if $p \in \{0, 1\}$ and $\epsilon_p = 1$ if $p \geq 2$.

Theorem 2.39. *For all $m \geq 0$ and $n \geq 0$, the number of weighted Γ_3-paths in $\mathcal{W}_{\Gamma_3}^{\Lambda_3}(n, -m)$ is equal to the number of non-weighted paths in $\mathcal{P}_{\Lambda_3}(n, -m)$. Specifically,*

$$|\mathcal{P}_{\Lambda_3}(n, -m)| = \sum_{\mu \in \mathcal{P}_{\Gamma_3}(n, -m)} w(\mu) = |\mathcal{W}_{\Gamma_3}^{\Lambda_3}(n, -m)|.$$

Proof. This follows from Theorem 2.10. □

Theorem 2.40. *For all $m \in \mathbb{Z}$ and $n \geq 1$, we have*

$$|\mathcal{F}_{\Lambda_3}(n, m)| = \sum_{k=0}^{\lfloor \frac{Nn-m}{N+K} \rfloor} \binom{n}{k} \binom{n(N+1) - k(N+K) - m}{n}, \tag{2.53a}$$

$$|\mathcal{P}_{\Lambda_3}(n, -1)| = \frac{1}{n} \sum_{k=0}^{\lfloor \frac{Nn+1}{N+K} \rfloor} \binom{n}{k} \binom{n(N+1) - k(N+K)}{n-1}. \tag{2.53b}$$

Proof. Applying Theorem 2.22 for the set $\Lambda_3 = \{V, U_N, D_K\}$, we see that $|\mathcal{F}_{\Lambda_3}(n, m)| = [x^{Nn-m}](1 + x^{N+K})^n / ((1 - x)^{n+1})$. Using Proposition 2.33, we obtain the formula for the number of free Λ_3-paths. Similarly, using Theorem 2.22, we obtain the second formula. □

Corollary 2.41. *For $n \geq 0$, if $K = 1$, then*

$$|\mathcal{P}_{\Lambda_3}(n,0)| = (-1)^n + \sum_{j=1}^{n} \sum_{k=0}^{\lfloor \frac{Nj+1}{N+1} \rfloor} \frac{(-1)^{n-j}}{j} \binom{j}{k} \binom{(N+1)(j-k)}{j-1}.$$

Proof. This follows from Proposition 2.24 and Corollary 2.40. □

Corollary 2.42. *For $m \geq 0$, let $P_m(x) = \sum_{n \geq 0} |\mathcal{P}_{\Lambda_3}(n,-m)| x^n$. We have*

$$\begin{aligned} P_0(x) &= 1 + xP_0(x) + xP_0(x) \sum_{k=1}^{N} \sum_{d=1}^{k} \binom{N-k+d}{d} \sum_{M} \prod_{j=1}^{d} P_{m_j}(x), \\ P_m(x) &= \delta_m x + x \sum_{k=0}^{N} \sum_{d=1}^{k+1} \binom{N-k+d-\epsilon_m}{N-k} \sum_{M} \prod_{j=1}^{d} P_{m_j}(x), \end{aligned} \tag{2.54}$$

where δ_m, M, and ϵ_p are defined in (2.43), (2.45), and (2.52), respectively.

Corollary 2.43. *If $N = K = 1$, then*

$$P_0(x) = \frac{1 - \sqrt{1-4x-4x^2}}{2x(1+x)} \quad P_1(x) = \frac{1 - 2x - \sqrt{1-4x-4x^2}}{2x}$$

Proof. This can be proved in much the same way as Corollary 2.31. We only note that the functional equation for $P_1(x)$ is now $0 = 2x + (2x-1)P_1(x) + xP_1(x)^2$ and $P_0(x) = 1/(1 - x - xP_1(x))$. □

Example. If $N = K = 1$, then

$$\begin{aligned} (|\mathcal{F}_{\Lambda_3}(n,0)|)_{n\geq 0} &= (1,2,8,32,136,592,2624,11776,53344,243392,\ldots) && (A006139) \\ (|\mathcal{F}_{\Lambda_3}(n,-1)|)_{n\geq 0} &= (1,4,16,68,296,1312,5888,26672,121696,\ldots) && (A179191) \\ (|\mathcal{P}_{\Lambda_3}(n,0)|)_{n\geq 0} &= (1,1,3,9,31,113,431,1697,6847,28161,117631,\ldots) && (A052709) \\ (|\mathcal{P}_{\Lambda_3}(n,-1)|)_{n\geq 0} &= (0,2,4,12,40,144,544,2128,8544,35008,145792,\ldots) && (A025227) \end{aligned}$$

The numbers starting with A denote corresponding sequences in OEIS [32].

Chapter 3

Raney paths and plane multitrees

This chapter is devoted to the study of plane multitrees. In Section 3.1, we introduce the concept of similar plane trees (resp. multitrees). In Section 3.2, we show that there is a bijection between the set $\mathcal{T}(n)$ of plane multitrees with n nodes and the set $\mathcal{R}(n)$ of Raney paths running from $(0, 1)$ to $(n, 0)$. In Section 3.3, we prove that there is a bijection between the set $\mathcal{R}_N(n)$ of N-Raney paths running from $(0, 1)$ to $(n, 0)$ and the family of $(N - 1, n, 1)$-Raney sequences. In Sections 3.4 – 3.7, we apply these two above-mentioned bijections to derive several combinatorial and statistical properties of plane multitrees.

3.1 Similar plane trees and multitrees

As we have already noted in Chapter 1, there are several equivalent definitions of plane trees in the literature (see the remark after Definition 1.6 on page 7). Most often they are defined as rooted unlabeled trees in which every internal node has additionally specified a liner order of its sons (see, e.g., Flajolet and Sedgewick [19]). This linear order is equivalent to an embedding of the tree in the plane.

It is worth pointing out that we define plane trees (see Definition 1.6) as rooted directed trees in which labels of vertices satisfy certain properties and these properties give an order of sons for every internal node. In this section, we define the concept of similar plane trees which shows that our definition of plane trees agrees with other definitions of plane trees that appear in the literature. However, first, we give some intuitions about plane trees.

Remarks. Let T be a plane tree such that $T = (V, E)$.

1. If we ignore directions of the arcs of T, then we obtain an undirected tree (acyclic and connected). Herein, we shall assume that the vertex 1 is the root of the plane tree and every arc $(u, v) \in E$ is going from the father u to its son v.
2. For every internal node $v \in V$, its sons are ordered. We will assume that these sons are ordered from left to right. This order is equivalent to an embedding of the tree in the plane, thus, we shall omit numbers of vertices on figures.
3. Recall that we denote by T_v the subtree of T rooted at the vertex v. If T_v is a subtree with exactly m vertices, then these vertices are in the set $\{x : v \leq x < v + m\}$.
4. For every $v \in V$, if v has exactly s sons $v_1 < v_2 < \cdots < v_s$, then for every $i \in \{1, \ldots, s-1\}$, the subtree T_{v_i} has the size $v_{i+1} - v_i$, all vertices of T_{v_i} are in the set $\{x : v_i \leq x < v_{i+1}\}$, the subtree T_{v_s} has the size $v + m - v_s$, and all vertices of T_{v_s} are in the set $\{x : v_s \leq x < v + m\}$, where m is the size of the tree T_v.
5. Suppose that we have a rooted undirected tree T and the sons of every vertex are ordered. Suppose that we change every edge $\{u, v\}$ where u is the father of v into the arc (u, v). If we number the vertices of such a modified tree using the depth first search algorithm, then we obtain a plane tree.

Definition 3.1. Let T_u and S_v be two subtrees of plane trees T and S, respectively. The trees T_u and S_v are *similar* if they have the same height h and

1. $h = 0$, or
2. $h > 0$ and for some s, we have
   (a) u has exactly s sons $u_1 < u_2 < \cdots < u_s$,
   (b) v has exactly s sons $v_1 < v_2 < \cdots < v_s$, and
   (c) for every $i \in \{1, \ldots, s\}$, the subtrees T_{u_i} and S_{v_i} are similar (see Figure 3.1).

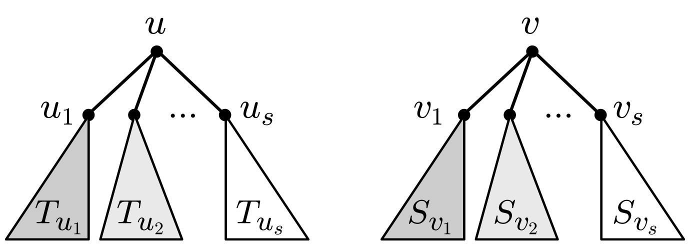

FIGURE 3.1: Two plane subtrees T_u (left) and S_v (right) with its sons.

Example. Let T be the left tree in Figure 3.2, and S be the right tree in Figure 3.2. Both roots of T and S have three sons, i.e., $2 < 5 < 6$ and $2 < 3 < 6$, respectively. However, the leftmost subtree T_2 of T is not similar to the leftmost subtree S_2 of S. Indeed, T_2 and S_2 have different height. Thus, T and S are not similar.

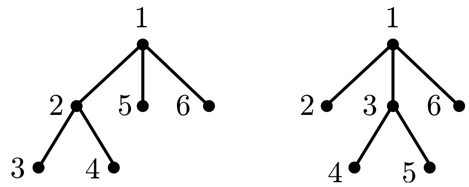

FIGURE 3.2: Two plane trees that are not similar.

Lemma 3.2. *Let T_u and S_v be two subtrees of plane trees T and S, respectively. If T_u and S_v are similar and $m = v - u$, then the function $h_m(x) = x + m$ is an isomorphism between T_u and S_v.*

Proof. The proof is by induction on the height h. If $h = 0$, then T_u and S_v have exactly one vertex and h_m maps u to v. Now, we assume that $h > 0$. First, observe that h_m maps u on v. Suppose that u has exactly s sons $u_1 < u_2 < \cdots < u_s$, and that v has exactly s sons $v_1 < v_2 < \cdots < v_s$. We shall show that for $1 \leq i \leq s$, h_m maps u_i to v_i.

Suppose that T_u has d vertices. These vertices are in the set $\{x : u \leq x < u + d\}$ (see the remarks at the beginning of this section). The leftmost son u_1 of u is $u + 1$ and the leftmost son v_1 of v is $v+1$, thus, h_m maps u_1 to v_1, as claimed. By (c) in Definition 3.1, for $1 \leq i \leq s$, the subtrees T_{u_i} and S_{v_i} are similar. Thus, by the induction hypothesis, we have $v_{i+1} - v_i = u_{i+1} - u_i$, which implies that T_{u_i} and S_{v_i} have the same number of nodes. Therefore, for $1 \leq i \leq s$, the function h_m maps u_i to v_i.

Finally, for $1 \leq i \leq s$, the function h_m maps the subtree T_{u_i} onto S_{v_i}, and, by the induction hypothesis, we see that h_m restricted to T_{u_i} is an isomorphism. Thus, the function h_m is an isomorphism between T_u and S_v. $\square$

Remark. It is clear that the following statement is also true. If h_m is an isomorphism between T_u and S_v, then T_u and S_v are similar.

Lemma 3.3. *If two plane trees $T = (V_T, E_T)$ and $S = (V_S, E_S)$ are similar, then $T = S$.*

Proof. In this case the identity function h_0 is an isomorphism. $\square$

Definition 3.4. Let T_u and S_v be two subtrees of plane multitrees T and S, respectively. The multitrees T_u and S_v are *similar* if they have the same height h and

1. $h = 0$, or
2. $h > 0$ and for some s, we have
   (a) u has exactly s sons $u_1 < u_2 < \cdots < u_s$,
   (b) v has exactly s sons $v_1 < v_2 < \cdots < v_s$, and

(c) for every $i \in \{1, \ldots, s\}$, the weight of the arc (u, u_i) is equal to the weight of (v, v_i), and the subtrees T_{u_i} and S_{v_i} are similar (see Figure 3.3).

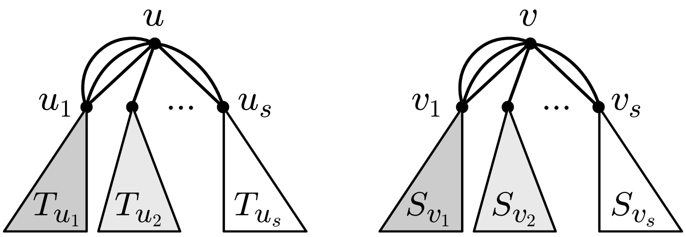

FIGURE 3.3: Two plane sub-multitrees T_u (left) and S_v (right) with its corresponding subtrees.

Lemma 3.5. *If two plane multitrees $T = (V_T, E_T, w_T)$ and $S = (V_S, E_S, w_S)$ are similar, then $T = S$.*

Proof. This can be proved in much the same way as Lemma 3.2 for $h_0(x)$. □

Definition 3.6. Let v be an internal node of a plane multitree T. Suppose that v has s sons $v_1 < v_2 < \cdots < v_s$ and $w((v, v_j)) = m_j$ for $j \in \{1, \ldots, s\}$. Let $\text{sons}_T(v)$ denote this list of the sons of v represented as follows:

$$\text{sons}_T(v) = (v_1^{m_1}, v_2^{m_2}, \ldots, v_s^{m_s}).$$

Example. Let T be the right tree given in Figure 3.2, we have $\text{sons}_T(1) = (2^1, 3^2, 6^1)$.

3.2 Bijection between Raney paths and plane multitrees

Recall that $\mathcal{T}(n)$ (resp. $\mathcal{T}_N(n)$) denotes the family of plane multitrees (resp. N-ary plane multitrees) with n nodes and $\mathcal{R}(n)$ (resp. $\mathcal{R}_N(n)$) denotes the family of Raney paths (resp. N-Raney paths) of length n.

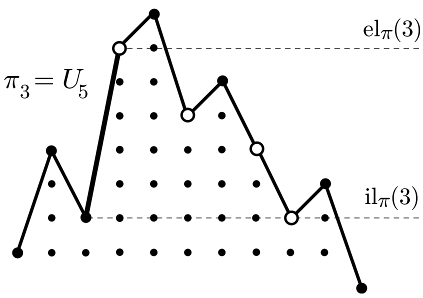

FIGURE 3.4: A 5-Raney path π running from $(0, 1)$ to $(10, 0)$. All the points of the list $\Pi_\pi(3)$ are marked using open circles. These points lie weakly between the initial $\text{il}_\pi(3)$ and ending $\text{el}_\pi(3)$ levels of π_3.

Definition 3.7. Let $\pi = \pi_1 \cdots \pi_n \in \mathcal{R}(n)$. Suppose that π_i is an up step U_k and that π_i connects two lattice points (u, l) and $(u+1, l+k)$. We denote by $\mathrm{il}_\pi(i)$ and $\mathrm{el}_\pi(i)$ the initial level $y = l$ and the ending level $y = l + k$, respectively. If $\mathrm{il}_\pi(i) = l$, then π passes through the points $(x_1, l+k), (x_2, l+k-1), \ldots, (x_{k+1}, l) \in \mathbb{R} \times \mathbb{Z}$ such that $x_1 < \cdots < x_{k+1}$ are chosen to be the leftmost ones, i.e., $x_j = \min\{x : x \geq i$ and π passes through $(x, k-j+1)\}$. Note that some of these points may not have the first coordinate integer. Let us remove them and denote by $\Pi_\pi(i)$ the list of the remaining points (both coordinates are integers).

Example. Let $\pi = U_3D_2U_5U_1D_3U_1D_2D_2U_1D_3$ (see Figure 3.4). We have $\pi_3 = U_5$, $\mathrm{il}_\pi(3) = 2$, $\mathrm{el}_\pi(3) = 7$, and $\Pi_\pi(3) = ((3,7),(5,5),(7,4),(8,2))$. The points of $\Pi_\pi(3)$ are marked using open circles in Figure 3.4.

Theorem 3.8. *For all $N \geq 0$ and $n \geq 1$, we have*

$$|\mathcal{R}_N(n)| = |\mathcal{T}_{N+1}(n)|. \tag{3.1}$$

Proof. For $n = 1$ and $N \geq 0$, we have $\mathcal{R}(1) = \mathcal{R}_N(1) = \{D_1\}$. Both $\mathcal{T}(1)$ and $\mathcal{T}_{N+1}(1)$ contain only one plane multitree with one node and zero edges, as claimed. To prove the assertion for $n \geq 2$, we shall

- define a map $\rho_n : \mathcal{R}(n) \to \mathcal{T}(n)$ (Definition 3.10 on page 52),
- define a map $\kappa_n : \mathcal{T}(n) \to \mathcal{R}(n)$ (Definition 3.14 on page 57),
- prove that ρ_n is the inverse function of κ_n (Lemma 3.18 on page 61), and
- prove that for every $N \geq 0$, the map ρ_n limited to $\mathcal{R}_N(n) \to \mathcal{T}_{N+1}(n)$ is a bijection (Corollary 3.19 on page 63).

□

To define maps between Raney paths and plane multitrees, we use an abstract data structure called *stack* which is understood as a list `S` of a finite number of objects $s_1, s_2, \ldots$ with two following operations: *push* an element a to `S` which adds the object a to `S` as the first element of `S`, and *pop* from `S` which returns and removes the first element in `S`. By the *top* of S we mean the first element of `S`. Let a, b be two different elements and $a \in$ `S`. We say that a *is above* b *in* `S` if $b \notin$ `S` or the index of a is smaller than the index of b in `S`.

Definition 3.9. Suppose that $T = (V, E, w)$ is a plane multitree such that $V = \{1, 2, \ldots, n\}$ for some $n \geq 1$ and in which (E, w) is the multiset of edges. Take $i \in V$ and suppose that there is an arc (i, j) in E and j is the maximal such number. By *joining the node i to its rightmost son in E* we mean adding the additional edge (i, j) to the multiset (E, w) of edges. In other words, we increase the weight $w((i, j))$ by one.

Example. Let $T = (V, E, w)$ be the left tree given in Figure 3.5. Joining 1 to its rightmost son in E adds the arc $(1, 5)$ to the multiset (E, w) of edges, or in other words, increases the value $w((1, 5))$ from 1 to 2. Joining 2 to its rightmost son in E adds the arc $(2, 3)$ to the multiset (E, w). The final tree after this two operations is the right tree given in Figure 3.5.

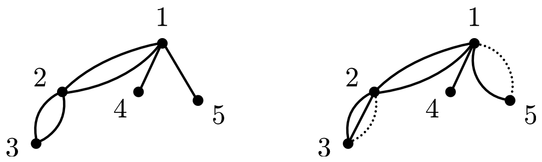

FIGURE 3.5: A plane multitree (left) and the plane multitree formed from the left one after the operation of joining nodes 1 and 2 to its rightmost son (right). The new additional edges are drawn using dotted lines.

Definition 3.10. The map $\rho_n : \mathcal{R}(n) \to \mathcal{T}(n)$ for $n \geq 2$.

Let $\pi = \pi_1 \cdots \pi_n \in \mathcal{R}(n)$, with $n \geq 2$. We specify the plane multitree $\rho_n(\pi) = (V, E, w)$ in n consecutive steps. First, we set $V = \{1, 2, \ldots, n\}$ and $E = \emptyset$. Let $\mathtt{S}_\rho$ be the empty stack of nodes.

Step 1. We have $\pi_1 = U_k$ for some $k \geq 0$. Push $k+1$ copies of the node 1 to $\mathtt{S}_\rho$ and set 1 to be the root of T.

Step i for $i \in \{2, 3, \ldots, n\}$. Pop the node from $\mathtt{S}_\rho$ and denote it by ν. Add the arc (ν, i) to E and set $w((\nu, i)) = 1$. We have two following cases:

(a) if $\pi_i = U_k$ for some $k \geq 0$, then push $k+1$ copies of the node i to $\mathtt{S}_\rho$,

(b) if $\pi_i = D_k$ for some $k \geq 1$, then pop $k-1$ nodes from $\mathtt{S}_\rho$ and join each of them to its rightmost son in E.

Remarks.

1. There is a one-to-one correspondence between the elements on the stack $\mathtt{S}_\rho$ and the edges of T. If an element is pushed to the stack $\mathtt{S}_\rho$, then it corresponds to the upper end of an edge. If an element ν is popped at the beginning of the ith step, then it forms the main edge (ν, i). This edge is added to E and $w((\nu, i))$ is set to 1. If an element ν is popped in part (b), then the additional edge (ν, j) is added to E and j is the rightmost son of ν (in the tree built so far). The edge (ν, j) already exists in E and the weight of (ν, j) is increased by one.
2. At the beginning of the ith step for $i \in \{2, 3, \ldots, n\}$, the vertex i is joined with its father which is the node just popped from the stack. If $\pi_i = U_k$, then $k+1$ copies of the node i are pushed to the stack. This determines that $\mathrm{odeg}(i) = k+1$ in the final tree. If $\pi_i = D_k$, then the node i is set to be a leaf and $k-1$ nodes are popped from the stack. These $k-1$ popped nodes form additional edges in the

tree. Moreover, they are added to the vertices on the path leading from the father of the node $(i+1)$ to i if $i < n$ or from the root to n if $i = n$ (see Lemma 3.11 given below).

Example. The four pictures (1)–(4) in Figure 3.6 present the first four steps of ρ_6 acting on the path $\pi = U_5 D_3 U_2 D_2 D_2 D_1$. In picture (i) for $i \in \{1,2,3,4\}$, we have the fragment $\pi_1 \cdots \pi_i$ of the path π (left on the figure) and the fragment of the tree $\rho_6(\pi)$ built so far (right on the figure). Solid lines in the trees represent edges of E, dotted arcs correspond to the content of the stack $\mathtt{S}_\rho$. The top of the stack is represented using an open circle. The numbers below the Raney path represent labels of nodes on the stack $\mathtt{S}_\rho$ and the numbers in circles are the elements on the top of the stack. The final plane multitree $\rho_6(\pi)$ is given in Figure 3.7.

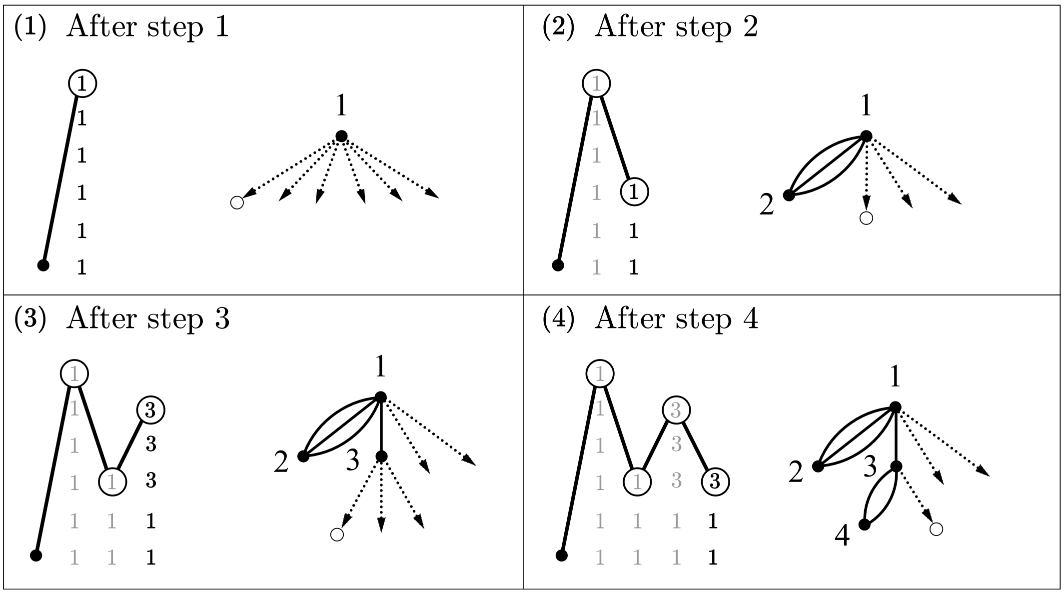

FIGURE 3.6: The first four steps of ρ_6 acting on the path $U_5 D_3 U_2 D_2 D_2 D_1$ (see Example given above for more details).

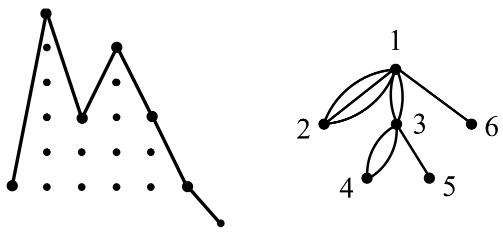

FIGURE 3.7: A 5-Raney path π (left) and the plane multitree $\rho_6(\pi)$ (right).

Lemma 3.11. *Let $n \geq 2$ and $1 \leq i \leq n$. Suppose that $\pi_1 \cdots \pi_i$ is a lattice path running from $(0,1)$ to (i,s). Suppose that (E,w) and $\mathtt{S}_\rho$ are the multiset of arcs and the stack, respectively, after the i steps of ρ_n acting on π.*

(i) The stack $\mathtt{S}_\rho$ contains s copies of nodes.

(ii) If $i < n$, then all nodes on the stack $\mathtt{S}_\rho$ lie on the path ψ leading from the root 1 *to the father of the node $i+1$ in the final tree $\rho_n(\pi)$. Moreover, these nodes lie on the path and the stack $\mathtt{S}_\rho$ in the same order and every edge of ψ has the weight equal to one in the tree built so far.*

(iii) The path leading from the root 1 *to the node i is the rightmost path in the tree built so far.*

(iv) The triple $(\{1, 2, \ldots, i\}, E, w)$ (i.e. the tree built up to the step i) is a plane multitree.

Proof. The proof is by induction on i. A simple verification shows that (i)–(iv) are satisfied for $i = 1$. Indeed, the path $\pi_1 = U_k$ begins at $(0, 1)$, ends at $(1, k+1)$, and the stack $\mathtt{S}_\rho$ contains only $k+1$ copies of the node 1. On the other hand, a tree with one node and zero arcs is a plane multitree. Let $i \in \{2, 3, \ldots, n\}$ and consider the ith step of $\rho_n(\pi)$.

(i) If $\pi_i = U_k$, then we pop one node from $\mathtt{S}_\rho$ and push exactly $k+1$ new nodes to $\mathtt{S}_\rho$. Therefore, the size of $\mathtt{S}_\rho$ is $s+k$. On the other hand, the path $\pi_1 \cdots \pi_i$ ends at level $s+k$, as claimed. If $\pi_i = D_k$, then we pop one plus $k-1$ nodes from $\mathtt{S}_\rho$. Thus, the total number of nodes in $\mathtt{S}_\rho$ is $s-k$. On the other hand, the path $\pi_1 \cdots \pi_i$ ends at level $s-k$, as claimed.

(ii), (iii) At the beginning of the ith step, we pop a node ν from $\mathtt{S}_\rho$, add the main arc (ν, i) to E, and set $w((\nu, i)) = 1$. If π_i is an up step, then we push some number of copies of the node i to $\mathtt{S}_\rho$. Before this operation, all nodes of $\mathtt{S}_\rho$ lay on the path leading from the root to the father of i, thus, now all nodes on the stack lie on the path leading from the root to i. In this case i will be the father of $i+1$ in the final tree. Moreover, we have not changed the weight of any arc and the path leading from the root to i is still the rightmost path in the constructed tree. If π_i is a down step, then we pop some number of nodes from the stack. The top of the stack is now the father of the node $i+1$ in the final tree. Thus, all remaining nodes on the stack lie on the path leading from the root to the father of $i+1$. Moreover, the joining operation possibly changed the weight of the arcs in the path leading from the father of $i+1$ to the node i, thus, the weight of the arcs in the path ψ was not changed, as claimed. Finally, the path from the root to i is the rightmost one in the tree constructed up to the step i.

(iv) At the beginning of the ith step, we pop a node ν from the stack and add an arc (ν, i). The stack contains nodes that labels are smaller than i and thus $\nu < i$. Moreover, as we have already shown, ν lies on the rightmost path of the tree, thus, the new added node i is added to the right of that path. Therefore, (i)–(iii) of Definition 1.6 hold and $T' = (\{1, 2, \ldots, i\}, E \cup \{(\nu, i)\}, w)$ is a plane multitree. If $\pi_i = U_k$, then we only add some nodes to the stack and the structure of the tree is not changed. Thus, T' is a plane multitree, as claimed. If $\pi_i = D_k$, then we pop $k-1$ nodes from the stack and join them to their rightmost sons in E. Because every node in $\{1, 2, \ldots, i\}$ has the father, the joining operation is well defined and it only changes the weight function w on the set of arcs E. Hence, the resulting tree i a plane multitree. □

Corollary 3.12. *For $n \geq 2$, if $\pi \in \mathcal{R}(n)$, then $\rho_n(\pi) \in \mathcal{T}(n)$.*

Proof. This follows from Lemma 3.11 (iv) for $i = n$. □

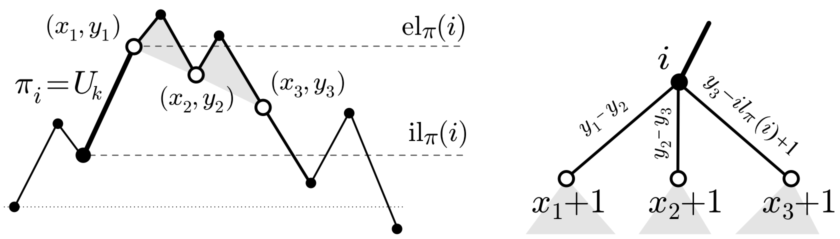

FIGURE 3.8: On the left: an up step $\pi_i = U_k$ in a Raney path π with the elements of $\Pi_\pi(i)$ drawn using open circles. The points of $\Pi_\pi(i)$ lie weakly between the initial $\mathrm{il}_\pi(i)$ and the ending $\mathrm{el}_\pi(i)$ levels (drawn using dotted lines) of the step π_i. On the right: a fragment of the multitree $\rho_n(\pi)$ with the vertex i and its sons.

Lemma 3.13. *Let $\pi = \pi_1 \cdots \pi_n \in \mathcal{R}(n)$, with $n \geq 2$. Let $T = (V, E, w) = \rho_n(\pi)$. If $\pi_i = U_k$ is an up step of π and $\Pi_\pi(i) = ((x_1, y_1), \ldots, (x_s, y_s))$, then*

$$\mathrm{sons}_T(i) = \big((x_1+1)^{y_1-y_2}, (x_2+1)^{y_2-y_3}, \ldots, (x_{s-1}+1)^{y_{s-1}-y_s}, (x_s+1)^{y_s-\mathrm{il}_\pi(i)+1}\big),$$

(see Figure 3.8).

Proof. By Lemma 3.11, the y-coordinate of the ending point of $\pi_1 \cdots \pi_i$ is equal to the size of the stack $\mathtt{S}_\rho$ after i steps of $\rho_n(\pi)$. The elements of the stack $\mathtt{S}_\rho$ are the indexes of up steps of π. Indeed, only in up steps we push new nodes to the stack. By Lemma 3.11 (i), at the beginning of the ith step, the stack has $\mathrm{il}_\pi(i)$ elements. In the ith step, the function ρ_n pops one element from $\mathtt{S}_\rho$ and pushes $k+1$ copies of the node i. Hence, after i steps of ρ_n, the size of the stack is $\mathrm{el}_\pi(i)$ and the first $k+1$ elements on the stack are the copies of the node i. The jth copy of the node i, for $1 \leq j \leq k+1$, will remain on the stack until it will be popped in the first step r of $\rho_n(\pi)$, with $r > i$, in which one of the two following conditions holds:

1. π_r is a step (up or down) that starts at the level $\mathrm{el}_\pi(i) - j + 1$. In this case, the copy of the node i is popped at the beginning of the rth step of $\rho_n(\pi)$ and corresponds to the main edge (i, r), and thus the node r will be a son of i. Moreover, the starting point of π_r is on the list $\Pi_\pi(i)$.

2. π_r is a down step that starts above the level $\mathrm{el}_\pi(i) - j + 1$ and ends below this level. In this case, the copy of the node i is popped in part b) of the rth step of $\rho_n(\pi)$ and corresponds to an additional edge (i, p), where p is the rightmost son of i in the constructed multitree so far, i.e., after these first r steps of $\rho_n(\pi)$.

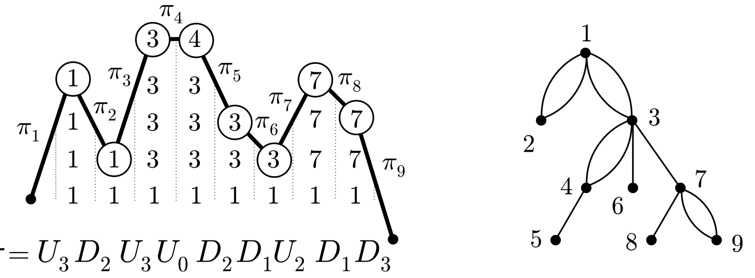

FIGURE 3.9: On the left: the Raney path π running from $(0, 1)$ to $(9, 0)$, the numbers below the path on the line $x = i$ represent labels of nodes on the stack $\mathtt{S}_\rho$ after i steps. The top of the stack is drawn using open circle. On the right: the final tree $\rho_9(\pi)$.

Therefore, the content of the stack $\mathtt{S}_\rho$ can be directly restored from the path (see Figure 3.9 and Example just below the proof). Namely, suppose that after i steps of $\rho_n(\pi)$, the stack $\mathtt{S}_\rho$ contains $\mathrm{el}_\pi(i)$ nodes $(v_1, v_2, \ldots, v_{\mathrm{el}_\pi(i)})$ and v_1 is on the top (see Figure 3.10). For each $j \in \{1, 2, \ldots, \mathrm{el}_\pi(i)\}$, we have

$$v_j = \max\left\{k : 1 \le k \le i, \pi_k \text{ is an up step, } \mathrm{il}_\pi(k) \le \mathrm{el}_\pi(i) - j + 1 \le \mathrm{el}_\pi(k)\right\}.$$

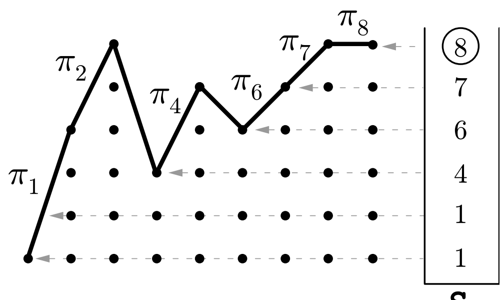

FIGURE 3.10: The stack $\mathtt{S}_\rho$ (right) after 8 steps of ρ_9 acting on the path $U_3U_2D_3U_2D_1U_1U_1U_0D_6$ (left). The top of the stack is drawn using an open circle.

The first $k+1$ nodes on the stack $\mathtt{S}_\rho$ are the copies of the node i and each of them will be popped in next steps of $\rho_n(\pi)$ in case (a) or in case (b) given above. Observe that the list of points $\Pi_\pi(i)$ contains exactly these lattice points that are the starting points of steps π_r (up or down) and for which the copy of the node i is popped in the case (a). For $j \in \{1, \ldots, s\}$, the point (x_j, y_j) in $\Pi_\pi(i)$ is the starting point of π_{x_j+1} and thus the node $(x_j + 1)$ will be a son of i in the final tree.

Hence, the list $\Pi_\pi(i)$ determines the list of sons of the node i in $\rho_n(\pi)$. Namely, because $\Pi_\pi(i) = ((x_1, y_1), \ldots, (x_s, y_s))$, we see that the node i will have exactly s sons $(x_1+1) < (x_2 + 1) < \cdots < (x_s + 1)$. Moreover, in the final tree $\rho_n(\pi)$, the weight of the arc

$(i, x_j + 1)$ will be equal to the difference $y_j - y_{j+1}$ for $j < s$ and $y_s + 1 - \mathrm{il}_\pi(i)$ for $j = s$ (see Figure 3.8). Indeed, in the $(x_1 + 1)$th step, we add the main edge $(i, x_1 + 1)$ and then in some steps t, with $x_1 + 1 \leq t \leq x_2$, we add $y_1 - y_2$ additional edges $(i, x_1 + 1)$, because $x_1 + 1$ is the rightmost son of i so far. In the $(x_2 + 1)$th step, we add the next main arc $(i, x_2 + 1)$ and then in some steps t, with $x_2 + 1 \leq t \leq x_3$, we add $y_2 - y_3$ additional edges $(i, x_2 + 1)$, because $x_2 + 1$ is the rightmost son of i so far. And so on, until all copies of the node i are popped from the stack. □

Example. Let $\pi = U_3U_2D_3U_2D_1U_1U_1U_0D_6$ (see Figure 3.9). We have $\pi_3 = U_3$ and $\Pi_\pi(3) = ((3,5),(5,3),(6,2))$. After the 2nd step of $\rho_9(\pi)$, the stack $\mathtt{S}_\rho$ contains exactly two copies of the node 1. In the 3rd step, the function $\rho_9(\pi)$ pops one node 1 and pushes 4 copies of the node 3 to the stack $\mathtt{S}_\rho$. Thus, after the 3rd step, the content of the stack is $(3,3,3,3,1)$ and 3 is on the top. The first copy of the node 3 (the top) will be popped at the beginning of the 4th step and it will correspond to the main edge $(3,4)$, the second copy of 3 will be popped in part b) of 5th step and it will correspond to the additional edge $(3,4)$, the third copy of 3 will be popped at the beginning of 6th step and it will correspond to the main edge $(3,6)$, and the last copy of 3 will be popped at the beginning of 7th step and it will correspond to the main edge $(3,7)$. Thus, $\mathrm{sons}_T(3) = (4^2, 6^1, 7^1)$, where $T = \rho_9(\pi)$.

Definition 3.14. The map $\kappa_n : \mathcal{T}(n) \to \mathcal{R}(n)$ for $n \geq 2$.

Let $T = (V, E, w) \in \mathcal{T}(n)$ and $V = \{1, 2, \ldots, n\}$, with $n \geq 2$. We specify $\kappa(T) = \pi = \pi_1 \cdots \pi_n$ in the following n steps. First, we set the beginning of π to be the lattice point $(0,1)$ and set $\mathtt{S}_\kappa$ to be the empty stack of nodes.

Step 1. We have $\mathrm{odeg}(1) = d$ for some $d \geq 1$. Set $\pi_1 = U_{d-1}$ and push the d sons (possibly repeated) of 1 to $\mathtt{S}_\kappa$ in order from right to left (leftmost son is on the top).

Step i for $i \in \{2, 3, \ldots, n\}$. Pop the node from $\mathtt{S}_\kappa$ and denote it by ν. We have two following cases:

(a) if $\mathrm{odeg}(\nu) = d$ and $d \geq 1$, then set $\pi_i = U_{d-1}$ and push all these d sons (possibly repeated) of ν to $\mathtt{S}_\kappa$ in order from right to left (leftmost son is on the top),

(b) if $\mathrm{odeg}(\nu) = 0$, then set $\pi_i = D_{r+1}$, where r is the number of nodes above $(i+1)$ in $\mathtt{S}_\kappa$ if $i < n$ or r is the number of all remaining nodes on the $\mathtt{S}_\kappa$ if $i = n$, and pop these r elements from $\mathtt{S}_\kappa$ (in the sequel we shall see that in such case if $i < n$, then the node $i + 1$ is on the stack).

Remarks.

(i) The map κ traverses the plane multitree T according to the depth-first search algorithm (DFS for short) and visits nodes of T according to their labels. Namely, in the ith step, $2 \leq i \leq n$, the popped node ν is exactly the node i (see Lemma 3.15 given below). Suppose that the root of T has s sons $v_1 < v_2 < \cdots < v_s$. First, DFS visits the root v of the tree and recursively traverses the subtree T_{v_1} rooted in v_1. After visiting all nodes of T_{v_1}, DSF recursively traverses the subtree T_{v_2} rooted in v_2, and so on. After visiting all subtrees of the root, DFS ends the search.

(ii) There is a one-to-one correspondence between the elements on the stack $\mathtt{S}_\kappa$ and the arcs of T. The nodes on the stack correspond to the lower ends of the arcs. In the first step, the nodes pushed to the stack correspond to the arcs outgoing from 1. As we have noted in (i), in the ith step, the popped node ν is exactly the node i. If the node i is not a leaf, then we push its outgoing edges to the stack. If i is a leaf, then we pop all nodes above $(i+1)$ in $\mathtt{S}_\kappa$ if $i < n$ or all remaining nodes if $i = n$, respectively. Moreover, these nodes lie on the path leading from the father of $(i+1)$ to i, or from the root to n if $i = n$ (see Lemma 3.15). Thus, they were visited in the previous steps and correspond to the additional edges.

(iii) The map κ_n changes every leaf into a down step and every internal node into an up step. Moreover, if $i \in V$ is an internal node of T, then the list $\mathrm{sons}_T(i)$ uniquely determines the list of points $\Pi_\pi(i)$ (see Lemma 3.17 given below for more details).

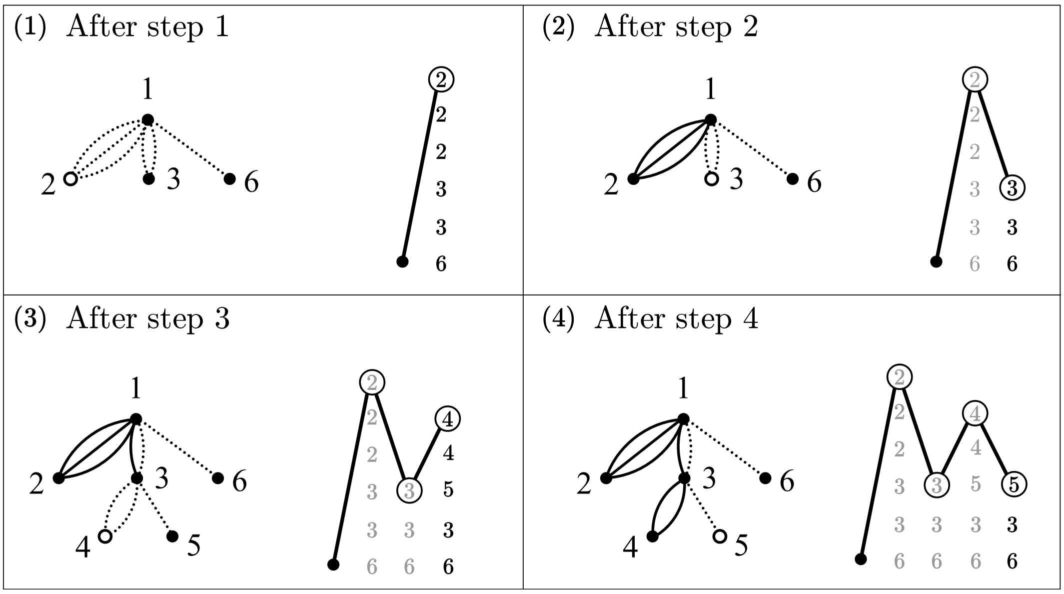

FIGURE 3.11: The first four steps of $\kappa_6(T)$ for the tree T given in Figure 3.7. Dotted lines correspond to the content of the stack $\mathtt{S}_\kappa$. The top of the stack is represented using an open circle. The numbers below the Raney path represent labels of nodes on the stack $\mathtt{S}_\kappa$ and the numbers in circles are the elements on the top of the stack. The complete Raney path $\kappa_6(T)$ is $U_5D_3U_2D_2D_2D_1$.

Example. The four pictures (1)–(4) in Figure 3.11 present the first four steps of κ_6 acting on the tree given in Figure 3.7.

Remark. Recall that Ω-path is a lattice path that consists of steps in the set $\Omega = \{(1,k) : k \in \mathbb{Z}\}$ (see Definition 1.1 on page 5). Note that a Raney path of length n is an Ω-path running from $(0,1)$ to $(n,0)$ in which only the ending point of the path lies below the line $y = 1$.

Lemma 3.15. *Let $n \geq 2$ and $1 \leq i \leq n$. Suppose that the stack $\mathtt{S}_\kappa$ after i steps of $\kappa(T)$ contains s nodes.*

(i) The path $\pi_1 \cdots \pi_i$ is an Ω-path running from $(0,1)$ to (i,s).

(ii) If $i < n$, then the top of the stack $\mathtt{S}_\kappa$ is the node $i+1$. Moreover, if $i+1$ is a leaf and $i+1 < n$, then the node $i+2$ is on the stack.

(iii) The stack $\mathtt{S}_\kappa$ contains only these nodes whose fathers lie on the path leading from the root to i, moreover, if $i < n$, then these nodes also lie on the path leading from 1 to the father of $i+1$. Moreover, they lie on the path and the stack in the same order, more precisely, if a path leading from the root to the father of v is shorter than the path from the root to the father of $u \neq v$, then u is nearer to the top of the stack than v does, if these paths have the same length, then u is nearer to the top if $u < v$.

(iv) The stack $\mathtt{S}_\kappa$ contains all sons u of the nodes of the path leading from the root to the node i such that $u > i$. The stack does not contain nodes that are to the left from the path.

Proof. The proof is by induction on i. A simple verification shows that the claim is true for $i = 1$. Indeed, if $\operatorname{odeg}(1) = s$, then the stack $\mathtt{S}_\kappa$ contains s elements. All these elements satisfy (ii)–(iv), the top is 2, and the path $\pi_1 = U_{s-1}$ starts at $(0,1)$ and ends at $(1,s)$, as claimed. Let $i \in \{2,3,\ldots,n\}$ and consider the ith step of κ.

(i) If $\operatorname{odeg}(i) = k$ and $k \geq 1$, then we pop one node, push k nodes, and set $\pi_i = U_{k-1}$. It follows that the stack contains now $s+k-1$ nodes and the path $\pi_1 \cdots \pi_i$ runs from $(0,1)$ to $(i, s+k-1)$, as claimed. If $\operatorname{odeg}(i) = 0$, then we pop $r+1$ nodes in total, where r is the number of nodes above $i+1$ on $\mathtt{S}_\kappa$, and we set $\pi_i = D_{r+1}$. It follows that the stack contains now $s-r-1$ nodes and $\pi_1 \cdots \pi_i$ runs from $(0,1)$ to $(i, s-r-1)$, as claimed.

(iii)-(iv) By the induction hypothesis, after the $(i-1)$th step, the fathers of nodes on $\mathtt{S}_\kappa$ lie on the path leading from 1 to $(i-1)$. The top of $\mathtt{S}_\kappa$ is now i and $\mathtt{S}_\kappa$ contains only these nodes whose fathers lie on the path leading from 1 to the father of i. Therefore, if $\operatorname{odeg}(i) > 0$, then we push all sons of i to $\mathtt{S}_\kappa$ from left to right, and the results follows. On the other hand, if $\operatorname{odeg}(i) = 0$, then we pop nodes above $(i+1)$ in $\mathtt{S}_\kappa$, which yields that the remaining nodes on $\mathtt{S}_\kappa$ also lie on the path leading from 1 to i and if $i < n$, then on the path leading from 1 to the father of $i+1$. Moreover, the sons of all vertices staying on the path leading from the root to the node i are on the stack.

(ii) If $\mathrm{odeg}(i) > 0$, then the leftmost child of i is $i+1$, and thus after the ith step, the top of the stack is $i+1$. If $\mathrm{odeg}(i) = 0$, then we pop all nodes above $i+1$ from $\mathtt{S}_\kappa$ and the top of the stack is now $i+1$. By the induction hypothesis, we see that the node $i+1$ is on the stack. In both cases, if $i+1$ is a leaf, then the structure of the plane multitree ensures that the node $i+2$ is a son of a node on the path leading from the root to the node i. Thus, from the property (iv), we see that $i+2$ is on the stack. □

Corollary 3.16. *For $n \geq 2$, if $T \in \mathcal{T}(n)$, then $\kappa_n(T) \in \mathcal{R}(n)$.*

Proof. The stack $\mathtt{S}_\kappa$ becomes empty only after the n steps of κ_n. By Lemma 3.15 for $i = n$, we see that $\kappa_n(T)$ is a Raney path running from $(0,1)$ to $(n,0)$ in which only the ending point lies below the line $y = 1$. □

Lemma 3.17. *Let $n \geq 2$ and $T = (V, E, w) \in \mathcal{T}(n)$, where $V = \{1, 2, \ldots, n\}$. Let $\pi = \pi_1 \cdots \pi_n = \kappa_n(T)$. If $i \in V$ is an internal node and $\mathrm{sons}_T(i) = (v_1^{m_1}, v_2^{m_2}, \ldots, v_s^{m_s})$, then*

$$\Pi_\pi(i) = \big((v_1 - 1, y_1), (v_2 - 1, y_2), \ldots, (v_s - 1, y_s)\big),$$

where $y_j = m_j + m_{j+1} + \cdots + m_s - 1 + \mathrm{il}_\pi(i)$ for $j \in \{1, \ldots, s\}$.

Proof. Suppose that after $i-1$ steps of κ_n, the stack $\mathtt{S}_\kappa$ contains exactly l nodes. By Lemma 3.15 (i), this number is the y-coordinate of the ending point of $\pi_1 \cdots \pi_{i-1}$. Thus, $l = \mathrm{il}_\pi(i)$. Because the node i is an internal one and because $\mathrm{odeg}(i) = m_1 + m_2 + \cdots + m_s$, we conclude that at the beginning of the ith step of κ_n, we pop the node i, push exactly $\mathrm{odeg}(i)$ possibly repeated sons of the node i, and set $\pi_i = U_{k-1}$, where $k = \mathrm{odeg}(i)$. Thus, $\pi_1 \cdots \pi_{i-1}\pi_i$ ends at $(i, l+k-1)$ and $\mathtt{S}_\kappa$ contains now $l+k-1$ nodes, where the first k ones are all the sons of the node i, i.e.,

$$\mathtt{S}_\kappa = (\underbrace{v_1, \ldots, v_1}_{m_1}, \underbrace{v_2, \ldots, v_2}_{m_2}, \cdots, \underbrace{v_s, \ldots, v_s}_{m_s}, \underbrace{\cdots}_{\text{the remaining } l-1 \text{ nodes}}).$$

Recall that every node on the stack corresponds to the lower end of an edge in E, and it can be popped in one of two following cases: a) at the beginning of the rth step, with $r > i$, as the main edge, and b) in the part (b) of the rth step, with $r > i$, as an additional edge. It is clear that for $j \in \{1, \ldots, s\}$, the first appearance of v_j on the stack $\mathtt{S}_\kappa$ corresponds to the main edge (i, v_j) and the remaining nodes v_j correspond to the additional edges (i, v_j) (see Figure 3.12).

By Lemma 3.15 (ii), the first appearance of the node v_j will be popped at the beginning of (v_j)th step, thus, after $(v_j - 1)$ steps of κ_n, the size of the stack will be equal to $y'_j = m_j + m_{j+1} + \cdots + m_s + l - 1$. Therefore, the path $\pi_1 \cdots \pi_{v_j - 1}$ will end at the point $(v_j - 1, y'_j)$. Moreover, this point is the first point on the level y'_j reached by the path $\pi_1 \cdots \pi_{v_j-1}$ weakly to the right of the line $x = i$, i.e.,

$$v_j - 1 = \min\{k : k \geq i, \text{ the path crosses the point } (k, y'_j)\}.$$

Thus, $(v_j - 1, y'_j)$ is the point of $\Pi_\pi(i)$. Moreover, the list $\Pi_\pi(i)$ contains only these s points determined by the first occurrences of $v_1, v_2, \ldots, v_s$. □

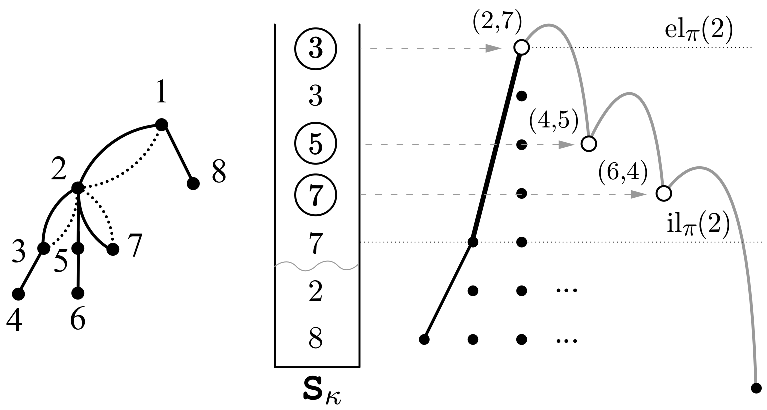

FIGURE 3.12: The stack S_κ (center) after 2 steps of κ acting on the plane multitree T (left) and points of $\Pi_\pi(2)$ determined by the list of $\mathrm{sons}_T(2)$. Additional edges are drawn using dotted lines. The first appearances of sons of the node 2 on the stack S_κ are drawn using open circles.

Example. Let $T = (V, E, w)$ be the plane multitree with 8 nodes given in Figure 3.12. The node 2 is an internal node and $\mathrm{sons}_T(2) = (3^2, 5, 7^2)$. The stack S_κ after 2 steps of κ contains 7 nodes $(3, 3, 5, 7, 7, 2, 8)$ (see the center of Figure 3.12) and the first 5 nodes of the stack determine $\Pi_\pi(2) = ((2, 7), (4, 5), (6, 4))$.

Let ψ be a path of a plane multitree and E_ψ be a set of edges of ψ. We set

$$\alpha(\psi) = \sum_{e \in E_\psi} (w(e) - 1). \tag{3.2}$$

Note that $\alpha(\psi)$ denotes the number of additional edges in ψ. For instance, we have $\alpha(\psi) = 3$, where ψ is the path leading from the root 1 to the node 4 in the tree given in Figure 1.10.

Lemma 3.18. *For $n \geq 2$, the map $\rho_n : \mathcal{R}(n) \to \mathcal{T}(n)$ is a bijection.*

Proof. We shall show that for any $\pi \in \mathcal{R}(n)$ and $T \in \mathcal{T}(n)$, we have (i) $\kappa_n(\rho_n(\pi)) = \pi$ and (ii) $\rho_n(\kappa_n(T)) = T$.

(i) Take $\pi = \pi_1 \cdots \pi_n \in \mathcal{R}(n)$, with $n \geq 2$. Let $T = (V, E, w) = \rho_n(\pi)$, with $V = \{1, 2, \ldots, n\}$ and $\mu = \mu_1 \cdots \mu_n = \kappa_n(\rho_n(\pi))$. We shall show that for $j \in \{1, \ldots, n\}$, we have $\pi_1 \cdots \pi_j = \mu_1 \cdots \mu_j$. The proof is by induction on j. Let $j = 1$. If $\pi_1 = U_k$, then $\mathrm{odeg}(1) = k + 1$ which implies $\mu_1 = U_k$, as claimed. Next, let $1 < j \leq n$ and assume that $\pi_1 \cdots \pi_{j-1} = \mu_1 \cdots \mu_{j-1}$. Suppose that $\pi_j = D_k$, the case where π_j is an up step is handled as π_1. Let us consider the jth step of ρ_n. We add a main edge (ν, j) to E, set $w((\nu, j)) = 1$, and set the node j to be a leaf in final tree $\rho_n(\pi)$. Next, we pop exactly $k - 1$ nodes from $\mathtt{S}_\rho$ and join each of them to their rightmost son in the constructed set of edges E. Note that these nodes lie on the path ψ leading from the father of the node $(j + 1)$ to the node j or from the root to the node i if $i = n$ in the final tree $\rho_n(\pi)$ (see Lemma 3.11). This joining operation increases the weight of arcs in the path ψ and whose weight before this operation were everywhere equal to one. Moreover, the weight of these arcs will not be changed in the further steps. Thus, $\alpha(\psi) = k - 1$ in the final tree $\rho_n(\pi)$.

On the other hand, let us consider the jth step of κ acting on $T = \rho_n(\pi)$. The node j is a leaf in T, therefore, we set $\mu_j = D_{r+1}$, where r is the number of nodes above the element $(j + 1)$ in $\mathtt{S}_\kappa$ if $j < n$ or the number of all remaining nodes in $\mathtt{S}_\kappa$ if $j = n$. Moreover, these nodes designate additional edges in the path ψ leading from the father of the node $(j + 1)$ to j or from the root to the node j if $j = n$. We pop these nodes from $\mathtt{S}_\kappa$. Now the top of the stack is the node $j + 1$ if $j < n$ or the stack is empty if $j = n$. Moreover, now, the stack $\mathtt{S}_\kappa$ does not contain any of nodes of the path ψ. Thus, $\alpha(\psi) = r$, $r = k - 1$ and $\mu_j = D_k$. From the above for $j = n$, we conclude that $\kappa_n(\rho_n(\pi)) = \pi$.

(ii) Let $T = (V_T, E_T, w_T) \in \mathcal{R}(n)$, with $V_T = \{1, 2, \ldots, n\}$. Let $R = (V_R, E_R, w_R) = \rho_n(\kappa_n(T))$, with $V_R = \{1, 2, \ldots, n\}$. First, observe that for every $i \in \{1, \ldots, n\}$, the node i is a leaf in T if and only if i is a leaf in R. Indeed, only leaves are mapped to down steps and vice versa. Similarly, the node i is an internal node in T if and only if i is an internal node in R. Finally, we must show that if i is an internal node, then $\mathrm{sons}_T(i) = \mathrm{sons}_R(i)$. Let $\pi = \pi_1 \cdots \pi_n = \kappa(T)$ and suppose that $\mathrm{sons}_T(i) = (v_1^{m_1}, v_2^{m_2}, \ldots, v_s^{m_s})$. First, by Lemma 3.17,

$$\Pi_\pi(i) = ((x_1, y_1), (x_2, y_2), \ldots, (x_s, y_s)),$$

where for $j \in \{1, \ldots, s\}$, we have $x_j = v_j - 1$ and $y_j = m_j + m_{j+1} + \cdots + m_s - 1$. Next, by Lemma 3.13, we obtain that $\mathrm{sons}_R(i)$ is given as follows:

$$\begin{aligned}
\mathrm{sons}_R(i) &= \left((x_1 + 1)^{y_1 - y_2}, (x_2 + 1)^{y_2 - y_3}, \ldots, (x_{s-1} + 1)^{y_{s-1} - y_s}, (x_s + 1)^{y_s + 1}\right) \\
&= \left(v_1^{m_1}, v_2^{m_2}, \ldots, v_{s-1}^{m_{s-1}}, v_s^{m_s}\right) \\
&= \mathrm{sons}_T(i).
\end{aligned}$$

Hence, $T = R$ and thus $T = \rho_n(\kappa_n(T))$. $\square$

Corollary 3.19. *For all $N \geq 0$ and $n \geq 2$, the map $\rho_n : \mathcal{R}_N(n) \to \mathcal{T}_{N+1}(n)$ is a bijection.*

Proof. Observe that for every up step U_k, the map ρ_n produces a node with outdegree $k+1$. Conversely, for every node with outdegree $k \geq 1$, the map κ_n produces an up step U_{k-1}. Therefore, every N-Raney path of length n is mapped to an $(N+1)$-ary plane multitree with n nodes and vice versa. $\square$

Now, we shall present a few useful properties of the bijection ρ_n that will play an essential role in the enumeration of plane multitrees in the next sections.

Definition 3.20. Take $\pi \in \mathcal{R}(n)$ and suppose that π has exactly m up steps $U_{u_1}, \dots, U_{u_m}$. Let $\epsilon(\pi)$ denote the sum $(u_1 + 1) + \dots + (u_m + 1)$.

Corollary 3.21. *Let $\pi \in \mathcal{R}(n)$ and $T = \rho_n(\pi)$.*

1. *The number of nodes with outdegree $d \geq 1$ in T equals the number of up steps U_{d+1} in the path π.*
2. *The number of internal nodes in T equals the number of all up steps in π.*
3. *The number of leaves in T equals the number of all down steps in π.*
4. *The number of edges in T equals $\epsilon(\pi)$.*

3.3 Raney sequences and Raney paths

In this section we show that there is a one-to-one correspondence between N-Raney paths and (m, n, d)-Raney sequences. Let $m, d \geq 1$ and $n \geq 0$, originally, an (m, n, d)-Raney sequence is a sequence of $mn + d$ integers in the set $\{1, 1 - m\}$ that sums to d, and its every partial sum is positive (see Graham et al. [21, p. 360]). For our purposes, we shall use a simple modification of this notation.

Definition 3.22. For $m, d \geq 1$ and $n \geq 0$, an (m, n, d)-*Raney sequence* is a sequence of $(mn + d)$ integers $a_1, a_2, \dots, a_{mn+d}$ satisfying

(i) $a_i \in \{-1, m - 1\}$ for $1 \leq i \leq mn + d$,

(ii) $a_1 + a_2 + \dots + a_{mn+d} = -d$, and

(iii) $a_1 + a_2 + \dots + a_i > -d$ for $1 \leq i < mn + d$.

Let $R(m, n, d)$ denote the number of all (m, n, d)-Raney sequences.

Example. An example of a $(3,2,1)$-Raney sequence is $(2,-1,2,-1,-1,-1,-1)$. An example of a $(4,2,3)$-Raney sequence is $(-1,3,-1,-1,-1,-1,3,-1,-1,-1,-1)$.

If $n=0$, then there is only one $(m,0,d)$-Raney sequence $(a_1,\ldots,a_d)=(-1,-1,\ldots,-1)$. If $n\geq 1$, then one can observe that every (m,n,d)-Raney sequence has exactly n terms valued by $m-1$ and $(m-1)n+d$ terms valued by -1. Using the Raney lemma (see Lemma 2.16 on page 30), we see that the number $R(m,n,1)$ of all $(m,n,1)$-Raney sequences is

$$R(m,n,1)=\frac{1}{mn+1}\binom{mn+1}{n} \qquad (m\geq 1, n\geq 0). \tag{3.3}$$

Indeed, having the family of all sequences of $mn+1$ integers in the set $\{-1,m-1\}$ that sum to -1, exactly $1/(mn+1)$ of them have all partial sums nonnegative (see Graham et al. [21, p. 360] for more details). This result can be generalized to the case of (m,n,d)-Raney sequences for any $d\geq 1$. Observe that an (m,n,d)-Raney sequence σ is the concatenation of exactly d sequences $\sigma^{(1)},\ldots,\sigma^{(d)}$ such that each $\sigma^{(i)}$ is an $(m,n_i,1)$-Raney sequence for some $n_i\geq 0$ and $n_1+\cdots+n_d=n$. It follows that there are exactly d cyclic shifts of any sequence of $mn+d$ integers in the set $\{-1,m-1\}$ that sums to $-d$ such that the property (iii) from Definition 3.22 is satisfied. Therefore (see [21, Eq. (7.70)]),

$$R(m,n,d)=\frac{d}{mn+d}\binom{mn+d}{n} \qquad (m\geq 1, d\geq 1, n\geq 0). \tag{3.4}$$

Lemma 3.23. *For $n\geq 1$ and $N\geq 0$, there is a bijection between $(N+1,n,1)$-Raney sequences and N-Raney paths of length n.*

Proof. Let $\sigma=(b_1,\ldots,b_{Nn+n+1})$ be an $(N+1,n,1)$-Raney sequence of integers in the set $\{-1,N\}$. The sum of σ is -1 and for any $1\leq i<Nn+n+1$, we have $b_1+\cdots+b_i\geq 0$. The sequence σ has exactly n terms equal to N. Moreover, $b_1=N$, therefore, we can partition σ into n subsequences $\sigma^{(1)},\ldots,\sigma^{(n)}$ in the following way:

$$\sigma=\Big(\underbrace{N,-1,\ldots,-1}_{\sigma^{(1)}},\underbrace{N,-1,\ldots,-1}_{\sigma^{(2)}},\ldots,\underbrace{N,-1,\ldots,-1}_{\sigma^{(n)}}\Big). \tag{3.5}$$

For $i\in\{1,\ldots,n\}$, let s_i denote the sum of $\sigma^{(i)}$. Observe that each $\sigma^{(i)}$ corresponds to the lattice step $(1,s_i)$ and this step is in $\{S_k=(1,k):k\leq N\}$. Because the property (iii) of Definition 3.22 is satisfied, the path $((1,s_1),\ldots,(1,s_n))$ is an N-Raney path of length n. On the other hand, every N-Raney path in $\mathcal{R}_N(n)$ can be represented as (3.5), and thus, corresponds to an $(N+1,n,1)$-Raney sequence. It is clear that this correspondence is one-to-one. □

Theorem 3.24. *For all $N \geq 1$ and $n \geq 1$, the number $T_N(n)$ of N-ary plane multitrees with n nodes is equal to*

$$T_N(n) = \frac{1}{Nn+1}\binom{Nn+1}{n} = \frac{1}{n}\binom{Nn}{n-1}. \tag{3.6}$$

Furthermore, $T_N(n)$ is equal to the number of $(N-1)$-Raney paths of length n.

Proof. By Theorem 3.8, for all $N \geq 1$ and $n \geq 1$, $|\mathcal{R}_{N-1}(n)| = |\mathcal{T}_N(n)| = T_N(n)$. On the other hand, by Lemma 3.23, $|\mathcal{R}_{N-1}(n)|$ is equal to the number of $(N, n, 1)$-Raney sequences. Using (3.3), we obtain the required formula. □

Example. Let us calculate the size of $\mathcal{T}_2(4)$. Using (3.6) yields

$$T_2(4) = \frac{1}{4}\binom{8}{3} = 14,$$

and all of these 2-ary plane multitrees with 4 nodes are given in Figure 1.11 on page 9. Independently of (3.6), it is easy to observe that for all $n \geq 1$ and $N \geq 1$, we have

$$T_1(n) = T_N(1) = 1, \qquad T_N(2) = N, \qquad T_N(3) = N^2 + \binom{N}{2}.$$

Lemma 3.25. *For all $0 \leq d \leq N$ and $n \geq 2$, the number of N-Raney paths of length n in which the first step is U_d is equal to*

$$\frac{N+d+1}{N(n-1)+d+1}\binom{(N+1)(n-1)+d-1}{n-2}. \tag{3.7}$$

Proof. Let A be the family of Raney paths running from $(0,1)$ to $(n,0)$ in which the first step is U_d. Take $\pi = \pi_1 \cdots \pi_n \in A$ and observe that $\pi' = \pi_2 \cdots \pi_n$ forms a Σ-path running from $(1, d+1)$ to $(n,0)$ in which $\Sigma = \{S_N, S_{N-1}, \ldots\}$ and only the ending point lies below the line $y = 1$. Let B be the family of $(N+1, n, d+1)$-Raney sequences in which the first term is N. Using the idea of the bijection from the proof of Lemma 3.23, we shall show that there is a one-to-one correspondence between A and B.

Namely, let us remove the first step in every path in A. Thus, A contains now the family of all Σ-paths running from $(1, d+1)$ to $(n,0)$ in which only the ending point lies below the line $y = 1$. Every path $\pi' \in A$ has $n-1$ steps $\pi_2, \ldots, \pi_n$ in Σ and can be represented as the sequence of steps $((1, s_2), (1, s_3), \ldots, (1, s_n))$, where each $s_i \in \{N, N-1, \ldots\}$. The sum $s_2 + s_3 + \cdots + s_n$ is $-d-1$ and every partial sum of $(s_2, s_3, \ldots_n)$, except the total sum, is greater than $-d-1$. Therefore, using the idea from the proof of Lemma 3.23, we show that π' can be represented as (3.5), and thus corresponds to an uniquely determined

Raney sequence in B. On the other hand, every Raney sequence in B corresponds to a path in A, and therefore, $|A| = |B|$.

To calculate the size of B, observe that the first term of any $(N+1, n-1, d+1)$-Raney sequence is either N or -1. The number of these $(N+1, n-1, d+1)$-Raney sequences in which the first term is -1 is $R(N+1, n-1, d)$. Indeed, if we remove this first -1, we obtain a Raney sequence that sums to $-d$. Thus, $|B| = R(N+1, n-1, d+1) - R(N+1, n-1, d)$. Using (3.4) and simplifying the result we obtain the required formula. □

3.4 Counting multitrees by leaves

Taking into account two previous sections we derive certain enumerating functions on plane multitrees. We start with the restriction on the number of leaves. For all $N \geq 1$ and $1 \leq k \leq n$, let $L_N(n, k)$ denote the number of N-ary plane multitrees with n nodes and exactly k leaves. For instance, $L_2(4, 2) = 6$ and all of such trees are given in Figure 3.13.

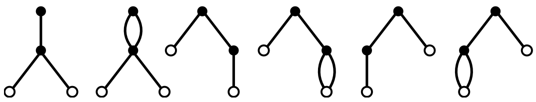

FIGURE 3.13: All 2-ary plane multitrees with 4 nodes and exactly 2 leaves.

Theorem 3.26. *For all $N \geq 1$ and $1 \leq k \leq n$, we have*

$$L_N(n, k) = \frac{1}{n}\binom{n}{k}\sum_{s=0}^{n-k}(-1)^s\binom{n-k}{s}\binom{N(n-k-s)}{n-1}. \tag{3.8}$$

Proof. By Corollary 3.21, the number of N-ary plane multitrees with k leaves is equal to the number of $(N-1)$-Raney paths with exactly k down steps. Similarly as in the proof of Lemma 3.23, we change an $(N-1)$-Raney path $((1, s_1), \ldots, (1, s_n))$ into $(N, n, 1)$-Raney sequence

$$\sigma = \big(\underbrace{N-1, -1, \ldots, -1}_{\sigma^{(1)}}, \underbrace{N-1, -1, \ldots, -1}_{\sigma^{(2)}}, \ldots, \underbrace{N-1, -1, \ldots, -1}_{\sigma^{(n)}}\big),$$

where s_i is the sum of the subsequence $\sigma^{(i)}$ for $i \in \{1, \ldots, n\}$. A down step in a Raney path uniquely corresponds to a subsequence $\delta = (N-1, -1, -1, \ldots, -1)$, where the number of (-1)'s is greater than $N-1$, in a Raney sequence. Let us treat the sequence σ as a placement of $(N-1)n+1$ indistinguishable balls (elements -1) into n distinguishable boxes (formed by $(N-1)$'s).

First, let A be the number of placements of $(N-1)n+1$ indistinguishable balls into n distinguishable boxes such that exactly k of them contain more than $N-1$ balls. Every such placement corresponds to an Σ-path running from $(0,1)$ to $(n,0)$ that has exactly k down steps and $\Sigma = \{S_{N-1}, S_{N-2}, \ldots\}$. By the Raney lemma (see Lemma 2.16 on page 30), we see that only $1/n$ of these placements correspond to the paths in which only the ending point lies below the line $y=1$. Therefore, $L_N(n,k) = A/n$.

To calculate A, we fix k boxes on $\binom{n}{k}$ ways and fill each of them by N balls. The remaining $((N-1)(n-k)-k+1)$ balls are placed into n boxes with the restriction that each of these $n-k$ unfixed boxes can contain at most $N-1$ balls. Now, we use the inclusion-exclusion principle. Let $A(i_1,\ldots,i_s)$ denote the number of ways to place $((N-1)(n-k)-k+1)$ balls into n boxes such that each of the boxes indexed by $i_1,\ldots,i_s$ contains at least N balls. The remaining balls $((N-1)(n-k)-k+1-sN)$ balls are placed in n boxes in all possible ways. Recall that the number of placements of a balls into b boxes in all possible ways is $\binom{a+b-1}{b-1}$. Therefore, $A(i_1,\ldots,i_s) = \binom{N(n-k-s)}{n-1}$. By the inclusion-exclusion principle,

$$A = \binom{n}{k}\sum_{s=0}^{n-k}(-1)^s \sum_{1\le i_1<\cdots<i_s\le n-k} A(i_1,\ldots,i_s),$$

and the result follows. $\square$

Example. The array $(L_2(i,j))_{i,j}$ for $1 \le i \le 8$ and $1 \le j \le 5$, is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 8 & 6 & 0 & 0 & 0 \\ 16 & 24 & 2 & 0 & 0 \\ 32 & 80 & 20 & 0 & 0 \\ 64 & 240 & 120 & 5 & 0 \\ 128 & 672 & 560 & 70 & 0 \end{pmatrix}.$$

This arrays is denoted by A091894 in OEIS [32].

Corollary 3.27. *For $N \ge 1$ and $1 \le k \le n$, the number of N-ary plane multitrees with k internal nodes is*

$$L_N(n,n-k) = \frac{1}{n}\binom{n}{k}\sum_{s=0}^{k}(-1)^s\binom{k}{s}\binom{N(k-s)}{n-1}. \tag{3.9}$$

3.5 Counting multitrees by edges

In this section, we consider another restriction on the family of N-ary plane multitrees with n nodes. Namely, we derive the number of these plane multitrees that have a fixed number of edges. For $N, n, s \geq 1$, let $E_N(n,s)$ denote the number of N-ary plane multitrees with n nodes and s edges. For instance, we have $E_2(4,4) = 6$ (see Figure 3.14).

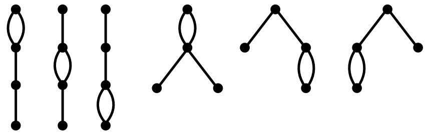

FIGURE 3.14: All 2-ary plane multitrees with 4 nodes and exactly 4 edges.

Theorem 3.28. *For all $N, n, s \geq 1$, we have*

$$E_N(n,s) = \frac{1}{n} \sum_{i=1}^{n-1} \sum_{j=0}^{i} (-1)^j \binom{n}{i} \binom{i}{j} \binom{s - jN - 1}{i-1}. \tag{3.10}$$

Proof. By Corollary 3.19, the number of N-ary plane multitrees with n nodes is the number of $(N-1)$-Raney paths of length n. By Corollary 3.21, we see that the number of edges in a plane multitree T with n nodes is equal to $\epsilon(\pi)$ (see Definition 3.20 on page 63), where $\pi = \rho_n(T)$ and ρ_n is a bijection from Section 3.2. Therefore, instead of counting N-ary plane multitrees with n nodes and s edges, we shall find the size of the family, denoted by A, of $(N-1)$-Raney paths π of length n for which $\epsilon(\pi) = s$.

Every $(N-1)$-Raney path has at least one and at most $(n-1)$ up steps. Therefore, we partition the family A into $n-1$ subsets $A_1, \ldots, A_{n-1}$ according to the number of up steps. Every path π in A_i has i up steps and $(n-i)$ down steps. There are $\binom{n}{i}$ ways to choose the indexes of these i up steps of π. Suppose that $U_{u_1}, \ldots, U_{u_i}$ are the up steps of π. Because $\epsilon(\pi) = s$, we have $u_1 + \cdots + u_i + i = s$ and each $u_j \in \{0, 1, \ldots, N-1\}$. The number of ways to fix these $u_1, \ldots, u_i$ is $\sum_{j=0}^{i} (-1)^j \binom{i}{j} \binom{s-jN-1}{i-1}$, and may be obtained by the exclusion-inclusion principle in much the same way as in the proof of Theorem 3.26. We only note that in this case, we have $(s-i)$ balls which we need to put in i boxes such that every box can contain at most $N-1$ balls.

Having fixed up steps, we need to fix down steps. Suppose that $D_{d_1}, \ldots, D_{d_{n-i}}$ are the down steps of $\pi \in A_i$. Observe that the sum $d_1 + \cdots + d_{n-i}$ is equal to $\epsilon(\pi) - i + 1 = s - i + 1$ and each d_j is a positive integer. Thus, the number of determining such sequences is $\binom{s-i}{n-i-1}$. Observe that not every placement of these up and down steps gives a Raney path in which only the ending point lies below the line $y = 1$. By the Raney lemma, only $1/n$ of these placements form Raney paths. Therefore,

$$|A_i| = \frac{1}{n}\binom{n}{i}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\binom{s-jN-1}{i-1}\binom{s-i}{n-i-1},$$

and $E_N(n,s) = |A_1| + \cdots + |A_{n-1}|$. □

Example. The array $(E_2(i,j))_{i,j\geq 0}$ is denoted by A091869 in OEIS [32]. Let us show the array $(E_3(i,j))_{i,j}$ for $1 \leq i,j \leq 8$,

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 5 & 12 & 15 & 13 & 6 & 3 \\ 0 & 0 & 0 & 13 & 40 & 64 & 64 & 49 \\ 0 & 0 & 0 & 0 & 36 & 135 & 255 & 320 \\ 0 & 0 & 0 & 0 & 0 & 104 & 456 & 1011 \\ 0 & 0 & 0 & 0 & 0 & 0 & 309 & 1554 \end{pmatrix}.$$

Corollary 3.29. *For $N \geq 1$ and $n \geq 1$, the number of N-ary plane trees (without additional edges) with n nodes is*

$$E_N(n,n-1) = \frac{1}{n}\sum_{i=1}^{n-1}\sum_{j=0}^{i}(-1)^j\binom{n}{i}\binom{i}{j}\binom{n-jN-2}{i-1}. \tag{3.11}$$

Remark. These numbers are considered by Takacs [34] and appear in OEIS [32] as the sequences A001006 ($N = 2$), A036765 ($N = 3$), A036766 ($N = 4$).

3.6 Counting nodes with specified outdegree

For $N \geq 1$, $n \geq 1$, and $0 \leq d \leq N$, let $G_N(n,d)$ denote the number of N-ary plane multitrees whose root has outdegree d, and let $M_N(n,d)$ denote the number of nodes of outdegree d in all N-ary plane multitrees with n nodes. For instance, $G_2(4,2) = 9$, and there are $M_2(4,2) = 21$ nodes with outdegree 2, $M_2(4,1) = 15$ nodes with outdegree 1, and $M_2(4,0) = 20$ nodes with outdegree 0 (leaves) in all 2-ary plane multitrees with 4 nodes (see Figure 3.15).

Theorem 3.30. *For $1 \leq d \leq N$ and $n \geq 2$, we have*

$$G_N(n,d) = \frac{N+d-1}{(N-1)(n-1)+d}\binom{N(n-1)+d-2}{n-2}, \tag{3.12}$$

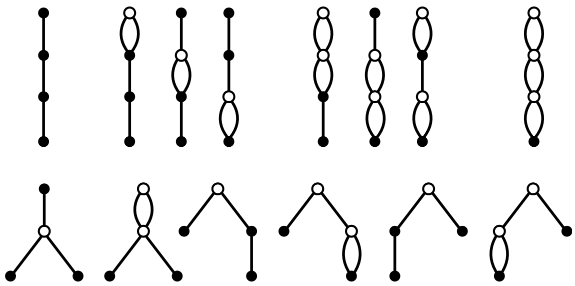

FIGURE 3.15: All 2-ary plane multitrees with four nodes. Nodes with outdegree 2 are drawn using open circles.

$G_N(1,0) = 1$, $G_N(1,d) = 0$ for $d \geq 1$, and $G_N(n,0) = 0$ for $n \geq 2$.

Proof. Let us consider the family of all N-ary plane multitrees with n nodes whose root has outdegree d. By Corollary 3.21, we see that the number of them is equal to the number of $(N-1)$-Raney paths of length n in which the first step is U_{d-1}. Using Lemma 3.25, we obtain the required formula. □

Theorem 3.31. *For $1 \leq d \leq N$ and $n \geq 2$, we have*

$$M_N(n,0) = \binom{N(n-1)}{n-1}, \tag{3.13a}$$

$$M_N(n,d) = \binom{N(n-1)+d-1}{n-2}, \tag{3.13b}$$

$M_N(1,0) = 1$, $M_N(n,0) = 0$ for $n \geq 2$, and $M_N(1,d) = 0$ for $d \geq 1$.

Proof. (a) Assume $d = 0$ and observe that $M_N(n,0) = \sum_{k=0}^{n} k\, L_N(n,k)$. Let us denote by $f(z)$ the power series $\sum_{n\geq 0}\sum_{k=0}^{n} k\, L_N(n,k) z^n$. With this notation, we have $M_N(n,0) = [z^n]f(z)$. To obtain the formula, we use the so-called Snake Oil Method developed by Wilf [41, Sec. 4.3] and apply the basic properties of the binomial coefficients. Relabeling $k \to n-k$, we obtain

$$\begin{aligned}
f(z) &= \sum_{n\geq 0}\sum_{k=0}^{n}\sum_{s=0}^{n-k}(-1)^s\binom{n-1}{n-k}\binom{n-k}{s}\binom{N(n-k-s)}{n-1}z^n \\
&= \sum_{n\geq 0}\sum_{k\geq 0}\sum_{s=0}^{k}(-1)^s\binom{n-1}{k}\binom{k}{s}\binom{N(k-s)}{n-1}z^n \\
&= \sum_{k\geq 0}\sum_{s=0}^{k}(-1)^s\binom{k}{s}\binom{N(k-s)}{k}\sum_{n\geq 0}\binom{N(k-s)-k}{n-1-k}z^n \\
&= \sum_{s\geq 0}\sum_{k\geq s}(-1)^{k-s}\binom{k}{s}\binom{Ns}{k}z^{k+1}(1+z)^{Ns-k}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{s \geq 0} \binom{Ns}{s} z^{s+1} (1+z)^{(N-1)s} \sum_{k \geq 0} (-1)^k \binom{(N-1)s}{k} z^k (1+z)^{-k} \\
&= \sum_{s \geq 0} \binom{Ns}{s} z^{s+1} (1+z)^{(N-1)s} \left(1 - \frac{z}{1+z}\right)^{(N-1)s} \\
&= \sum_{s \geq 0} \binom{Ns}{s} z^{s+1}.
\end{aligned}$$

(b) Fix $d \in \{1, 2, \ldots, N\}$. To calculate the number of nodes of outdegree d in all N-ary plane multitrees of n nodes, observe that any such node in a plane multitree T uniquely corresponds to an up step U_{d-1} in the path $\pi = \rho_n(T)$ under the bijection from Section 3.2. Therefore, the required number is equal to the number of all occurrences of the step U_{d-1} in all $(N-1)$-Raney paths of length n.

We apply Theorem 2.19 from Chapter 2 to obtain this number. First, we set Σ to be $\{S_{N-1}, S_{N-2}, \ldots\}$. Observe that there is a simple one-to-one correspondence between the family of 1-primary Σ-paths in $\mathcal{P}_\Sigma(1, n)$ (see Definition 1.2 on page 6) and the family $\mathcal{R}_{N-1}(n)$ of $(N-1)$-Raney paths of length n. Therefore, by Theorem 2.19, the number of steps U_{d-1} in all paths of $\mathcal{R}_{N-1}(n)$ is $|\mathcal{F}_\Sigma(d, n-1)|$, where $\mathcal{F}_\Sigma(d, n-1)$ is the family of free Σ-paths running from $(0,0)$ to $(n-1, -d)$ (see Definition 1.3 on page 6). Finally, by Theorem 2.22, we obtain

$$|\mathcal{F}_\Sigma(d, n-1)| = [y^{(N-1)(n-1)+d}] \Big(\sum_{S_k \in \Sigma} y^{N-1-k} \Big)^{n-1} = [y^{(N-1)(n-1)+d}] \frac{1}{(1-y)^{n-1}}.$$

Using the binomial theorem, the result follows. □

3.7 Statistical properties of plane multitrees

In this section we derive some statistical properties of plane multitrees. Namely, for $N \geq 1$ and $n \geq 1$, let us consider the family of all nodes in all N-ary plane multitrees with n nodes. The size of this family is $nT_N(n)$. For $0 \leq d \leq N$, let $B(N, n, d)$ denote the ratio of the number of these nodes with outdegree d to the number of all nodes, i.e.,

$$B(N, n, d) = \frac{M_N(n, d)}{nT_N(n)}. \tag{3.14}$$

Note that $nB(N, n, d)$ is the expected number of nodes with outdegree d in an N-ary plane multitree of n nodes. Next, for $N \geq 1$ and $n \geq 1$, let $J(N, n)$ denote the expected outdegree of a node in an N-ary plane multitree of n nodes, i.e.,

$$J(N,n) = \frac{1}{nT_N(n)} \sum_{d=0}^{N} dM_N(n,d). \tag{3.15}$$

Corollary 3.32. *For all $n \geq 2$ and $1 \leq d \leq N$, we have*

$$B(N,n,0) = \frac{(Nn-N)^{\underline{n-1}}}{(Nn)^{\underline{n-1}}}, \tag{3.16a}$$

$$B(N,n,d) = (n-1)\frac{(Nn-1-N+d)^{\underline{n-2}}}{(Nn)^{\underline{n-1}}}, \tag{3.16b}$$

where $a^{\underline{m}} = a(a-1)\cdots(a-m+1)$ for $m \geq 1$, and $a^{\underline{0}} = 1$.

Proof. The formulas directly follow from Theorem 3.24 and Theorem 3.31. □

Remark. Dershowitz and Zaks [8] showed that the expected number of leaves in a plane tree (without additional edges) with n nodes is $n/2$. By (3.16a), the expected number of leaves in an N-ary plane multitree with n nodes is

$$\frac{(Nn-N)^{\underline{n-1}}}{N(Nn-1)^{\underline{n-2}}}. \tag{3.17}$$

Lemma 3.33. *For $1 \leq d \leq N$ and $n \geq 1$, we have*

$$\lim_{n\to\infty} B(N,n,0) = \left(1 - \frac{1}{N}\right)^{N}, \tag{3.18a}$$

$$\lim_{N\to\infty} B(N,n,0) = \left(1 - \frac{1}{n}\right)^{n-1}, \tag{3.18b}$$

$$\lim_{n\to\infty} B(N,n,d) = \frac{1}{N}\left(1 - \frac{1}{N}\right)^{N-d}, \tag{3.18c}$$

$$\lim_{N\to\infty} B(N,n,d) = 0. \tag{3.18d}$$

Proof. We show (3.18a). For $n \geq N+2$, we can reduce the fraction (3.16a) and rewrite it as $(Nn-n+1)^{\underline{N}}((Nn)^{\underline{N}})^{-1}$. Next, we extract n from every term of products in nominator and denominator to get

$$\frac{n^N}{n^N}\frac{(N-1+1/n)(N-1+0/n)\cdots(N-1-(N-2)/n)}{N(N-1/n)\cdots(N-(N-1)/n)}.$$

If $n \to \infty$, then the above tends to $((N-1)/N)^N$, as claimed. The same method works for the other limits. □

Theorem 3.34. *We have*

$$\lim_{N\to\infty}\lim_{n\to\infty} B(N,n,0) = \lim_{n\to\infty}\lim_{N\to\infty} B(N,n,0) = \frac{1}{e}, \tag{3.19}$$

where e stands here for the base of the natural logarithm.

Proof. This directly follows from Lemma 3.33. We only need to recall that

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{n+b} = e^a,$$

for any real numbers a, b. □

Theorem 3.35. *For $N \geq 1$, we have*

$$\lim_{n\to\infty} J(N,n) = N\left(1-\frac{1}{N}\right)^{N+1} + 1. \tag{3.20}$$

Proof. Applying (3.6) and (3.13b) in (3.15), we obtain

$$J(N,n) = \frac{(n-1)}{(Nn)^{\underline{n-1}}}\sum_{d=1}^{N} d\cdot(Nn-N+d-1)^{\underline{n-2}}.$$

For $n \geq N+2$, the falling factorials in the sum share common terms. A simple calculation yields that the dth summand can be rewritten as $d\cdot(Nn-N+d-1)^{\underline{d-1}}(Nn-N)^{\underline{n-N-1}}(Nn-n+1)^{\underline{N-d}}$. If we divide $(Nn-N)^{\underline{n-N-1}}$ by $(Nn)^{\underline{n-1}}$, then we obtain $1/(Nn)^{\underline{N}}$. Therefore,

$$J(N,n) = \frac{(n-1)}{(Nn)^{\underline{N}}}\sum_{d=1}^{N} d\cdot(Nn-N+d-1)^{\underline{d-1}}(Nn-n+1)^{\underline{N-d}}.$$

Extracting n from every term and taking the limit of the result, we obtain the sum of N geometric progressions

$$\lim_{n\to\infty} J(N,n) = \frac{(N-1)^N}{N^{N+1}}\sum_{d=1}^{N} d\left(\frac{N}{N-1}\right)^d.$$

Using standard methods, we obtain the required formula. □

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