Binomial coefficients and generating functions - Shortcuts

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1 Binomial coefficients

By the *binomial coefficient* we mean the symbol

$$\binom{n}{k}$$

which has several equivalent definitions.

1. **Binomial theorem**. Binomial coefficient as the coefficient in the following power series expansion
$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}, \qquad n \in \mathbb{Z} \text{ and } n \geq 0. \tag{1}$$

2. **Combinatorial interpretation**. For $n, k \in \mathbb{Z}$ such that $0 \leq k \leq n$, $\binom{n}{k}$ is equal to the number of subsets X of the set $\{1, 2, \ldots, n\}$ such that $|X| = k$.

3. **Exact formula**. For nonnegative $n, k \in \mathbb{Z}$ such that $0 \leq k \leq n$,
$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n^{\underline{k}}}{k!} \tag{2}$$
where $0! = n^{\underline{0}} = 1$, and $n! = 1 \cdot 2 \cdots n$, and $n^{\underline{k}} = n \cdot (n-1) \cdots (n-k+1)$.

4. **Recurrence relation**. For nonnegative $n, k \in \mathbb{Z}$ such that $0 \leq k \leq n$,
$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \tag{3}$$
and $\binom{n}{0} = \binom{n}{n} = 1$.

2 Properties of the binomial coefficients

1. **Trinomial revision**:
$$\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{a-b} \tag{4}$$

2. **Upper negation**:
$$\binom{n}{k} = (-1)^k \binom{k-n-1}{k} \tag{5}$$

3. **Absorption**:

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1} \tag{6}$$

4. **Parallel summation**:

$$\sum_{k=0}^{n}\binom{r+k}{k} = \binom{r+n+1}{n} \tag{7}$$

5. **Upper summation**:

$$\sum_{k=0}^{n}\binom{k}{m} = \binom{n+1}{m+1} \tag{8}$$

6. **Vandermonde convolution**:

$$\sum_{k=0}^{n}\binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n} \tag{9}$$

7. **Other**:

$$\binom{n-1}{k} - \binom{n-1}{k-1} = \frac{n-2k}{n}\binom{n}{k} \tag{10}$$

$$\binom{n}{k} = \frac{n+1-k}{k}\binom{n}{k-1} \tag{11}$$

From [2]:

$$\sum_{\lambda\geq 0}\binom{A}{\lambda}\binom{B+\lambda}{C}(-1)^{\lambda} = (-1)^{A}\binom{B}{C-A}, \qquad A,B,C\geq 0 \tag{12}$$

$$\binom{M}{p}\binom{N}{q} = \sum_{\lambda\geq 0}\binom{M-N+q}{p+q+\lambda}\binom{\lambda}{q}\binom{N}{\lambda} \tag{13}$$

From [1, Eq. 5.28]

$$\sum_{k}\binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r+k}{m+n} = \binom{r}{m}\binom{s}{n} \tag{14}$$

$$\sum_{0\leq k\leq l}\binom{l-k}{m}\binom{q+k}{n} = \binom{l+q+1}{m+n+1} \tag{15}$$

$$\binom{-1/2}{k} = (-1)^{k}\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2^{k}k!}\frac{2\cdot 4\cdot 6\cdots 2k}{2^{k}k!} \tag{16}$$

$$= (-1)^{k}\frac{1}{2^{2k}}\binom{2k}{k} \tag{17}$$

$$\binom{1/2}{k} = (-1)^{k+1}\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2^{k}k!}\frac{2\cdot 4\cdot 6\cdots(2k)}{2^{k}k!} \tag{18}$$

$$= (-1)^{k-1}\frac{1}{2^{2k}(2k-1)}\binom{2k}{k} \tag{19}$$

$$[x^{n}]\frac{x^{k}}{(1-x)^{k+1}} = \binom{n}{k} \tag{20}$$

3 Generating function - some expansions

$$(1+x)^\lambda = \sum_{k\geq 0}\binom{\lambda}{k}x^k, \qquad \lambda\in\mathbb{R} \tag{21}$$

$$(\alpha+\beta x+\gamma x^2)^n = \sum_{k\geq 0}\sum_{j=0}^{k}\binom{n}{j}\binom{j}{k-j}\alpha^{n-j}\beta^{2j-k}\gamma^{k-j}x^k \tag{22}$$

$$\frac{1}{(1-x)^n} = \sum_{k\geq 0}\binom{n+k-1}{k}x^k = \sum_{k\geq 0}\binom{n+k-1}{n-1}x^k \tag{23}$$

$$((1+z)^{N+2}-1)^n = \sum_{l\geq 0}\sum_{s=0}^{n}(-1)^s\binom{n}{s}\binom{(N+2)(n-s)}{l}z^l \tag{24}$$

$$\left(\frac{1-z^{N+2}}{1-z}\right)^n = \sum_{k\geq 0}\sum_{s=0}^{\lfloor\frac{k}{N+2}\rfloor}(-1)^s\binom{n}{s}\binom{n-1+k-s(N+2)}{n-1}z^k \tag{25}$$

$$\frac{(1-z^{N+2})^n}{(1-z)^{2n+1}} = \sum_{k\geq 0}\sum_{s=0}^{\lfloor\frac{k}{N+2}\rfloor}(-1)^s\binom{n}{s}\binom{2n+k-s(N+2)}{2n}z^k \tag{26}$$

$$\frac{1}{(1-z-z^2)^n} = \sum_{k\geq 0}\sum_{j=0}^{k}\binom{n+j-1}{j}\binom{j}{k-j}z^k \tag{27}$$

$$= 1+7z+35z^2+140z^3+490z^4+1554z^5+O(z^6) \tag{28}$$

$$\sqrt{1-6z-3z^2} = \sum_{n\geq 0}\frac{1}{2^n}\sum_{s\geq 0}\frac{3^s}{(2s-1)}\binom{2s}{s}\binom{s}{n-s} \tag{29}$$

$$= 1-3z-6z^2-18z^3-72z^4-324z^5-1566z^6+O(z^7) \tag{30}$$

4 Compositions

Let $C(n,k,a,b)$ be the number of vector solutions

$$i_1+\cdots+i_n = k, \qquad a\leq i_1,i_2,\ldots,i_n\leq b$$

Then

$$C(n,k,0,\infty) = [z^k](1+z+z^2+\cdots)^n = \binom{n+k-1}{n-1}, \tag{31}$$

$$C(n,k,1,\infty) = [z^k](z+z^2+z^3+\cdots)^n = \binom{k-1}{n-1}, \tag{32}$$

$$C(n,k,0,N) = [z^k](1+z+\cdots+z^N)^n = [z^k]\frac{(1-z^{N+1})^n}{(1-z)^n} \tag{33}$$

$$= \sum_{s=0}^{\lfloor\frac{k}{N+1}\rfloor}(-1)^s\binom{n}{s}\binom{n+k-1-(N+1)s}{n-1} \tag{34}$$

$$= ???\ \text{by inclusion} - \text{exclusion principle} \tag{35}$$

5 Formal power series

Let $F(z) = f_0 + f_1 z + f_2 z^2 + \cdots$ be a formal power series. Then

$$[z^n] z^d F(z) = [z^{n-d}] F(z) = f_{n-d} \tag{36}$$

$$[z^n] F(z)^k = \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\ldots,i_k \geq 0}} f_{i_1} f_{i_2} \cdots f_{i_k}. \tag{37}$$

If $F(0) = 0$, that is, $f_0 = 0$, then

$$[z^n] F(z)^k = \sum_{\substack{i_1+\cdots+i_k=n \\ i_1,\ldots,i_k \geq 1}} f_{i_1} f_{i_2} \cdots f_{i_k}. \tag{38}$$

Fact 1 *A formal power series $f = \sum_{n\geq 0} a_n x^n$ has a* reciprocal $1/f$, *i.e., such that $f * 1/f = 1$ (convolution), if and only if $a_0 \neq 0$.*

Proof. Let $1/f = \sum_{n\geq 0} b_n x^n$. Then $f \cdot (1/f) = 1$. Therefore, $a_0 b_0 = 1$. Further,

$$b_n = \frac{-1}{a_0} \sum_{k\geq 1} a_k b_{n-k}.$$

This determines $b_1, b_2, \ldots$ uniquely, as claimed. ∎

Definition 1 *The* inverse *of a series f, if it exists, is a series g such that*

$$f(g(x)) = g(f(x)) = x.$$

If $f = \sum_{n\geq 0} a_n x^n$ then $f(g(x))$ means

$$f(g(x)) = \sum_{n\geq 0} a_n g(x)^n.$$

Fact 2 *The composition $f(g(x))$ of two formal power series is defined if and only if $g_0 = 0$ or f is a polynomial. We claim that if $f(0) = 0$ the inverse series exists if and only if the coefficient of x is nonzero in the series f.*

Logarithmic differentiation

$$\frac{d}{dz} \log(F_n(z)) = \frac{1}{F_n(z)} \frac{d}{dz} F_n(z).$$

$$\left(\sum_{k\geq 0} f_{n,k} z^n \right) \cdot \frac{d}{dz} \log(F_n(z)) = \sum_{k\geq 0} (k+1) f_{n,k+1} z^n.$$

Lagrange Inversion Formula [3]

Let $f(u)$ and $\phi(u)$ be formal power series in u, with $\phi(0) = 1$. Then there is a unique formal power series $u = u(t)$ that satisfies $u = t\phi(u)$. Further, the value $f(u(t))$ of f at the root $u = u(t)$, when expanded in a power series in t about $t = 0$, satisfies

$$[t^n]\{f(u(t))\} = \frac{1}{n} [u^{n-1}]\{f'(u) \phi(u)^n\}. \tag{39}$$

6 Other

Apart. Rewrites a rational expression as a sum of terms with minimal denominators.

$$\frac{1}{(x-x_1)(x-x_2)} = \frac{1}{x_1-x_2}\left(\frac{1}{x_2(1-x/x_2)} - \frac{1}{x_1(1-x/x_1)}\right)$$
$$\frac{x}{(x-x_1)(x-x_2)} = \frac{1}{x_1-x_2}\left(\frac{1}{1-x/x_2} - \frac{1}{1-x/x_1}\right)$$

References

[1] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. *Concrete Mathematics*. Addison-Wesley Publishing Company, second edition, 1994.

[2] D. E. Knuth. *The Art of Computer Programming*, volume vol. 1: Fundamental Algorithms. Addison-Wesley, third edition, 1997.

[3] Herbert S. Wilf. *Generatingfunctionology*. Academic Press, 1994.