PL ISSN 0459-6854

B U L L E T I N

DE LA SOCIÉTÉ DES SCIENCES ET DES LETTRES DE ŁÓDŹ

2009 Vol. LIX

Recherches sur les déformations Vol. LIX, no. 1

pp. 103–116

Andrzej Krzysztof Kwaśniewski

ON COMPOSITIONS OF NUMBERS AND GRAPHS

Summary

The main purpose of this note is to pose a couple of problems which are easily formulated thought some seem to be not yet solved. These problems are of general interest for discrete mathematics including a new twig of a bough of theory of graphs i.e. related with given graph compositions. The problems result from and are served in the entourage of series of exercises with hints based predominantly on Knopfmacher et all. recent papers.

1. Number compositions

> *"The number of compositions of n into k parts – all* ***distinct = ?*** *...*
> *that is the question ..."*

Let us start by recalling indispensable notions thus establishing notation and terminology. Following Clark Kimberling [1] we define natural number compositions and their family cardinality $C(n, k, a, b)$ as follows.

Definition 1.1 The composition of the natural number n is the vector

$$\langle i_1, i_2, ..., i_k \rangle$$

solution of the Diophantine equation

$$(D^*) \qquad i_1 + i_2 + ..., i_k = n, \quad a \le i_1, i_2, ..., i_k \le b.$$

$C(n, k, a, b) =$ the number of vector solutions of (D^*)

Exercise 1.1. Consider $C(n, k, 0, \infty)$ and $C(n, k, 1, \infty)$. Show that

$$C(n, k, 0, \infty) = \binom{n+k-1}{k-1}, \qquad C(n, k, 1, \infty) = \binom{n-1}{k-1}$$

Hint. As for $C(n,k,0,\infty)$ consider below the obvious Manhattan paths coding by vector solutions of $(D*)$. Naturally $C(n,k,0,\infty)$ = number of Manhattan paths from A to B covered according to the rule:
n steps **right** $\rightarrow$ and $k-1$ steps **up** $\uparrow$
(there are k levels or k floors or k storeys)
with coding: i_s = number of steps right at the s-th level (floor)

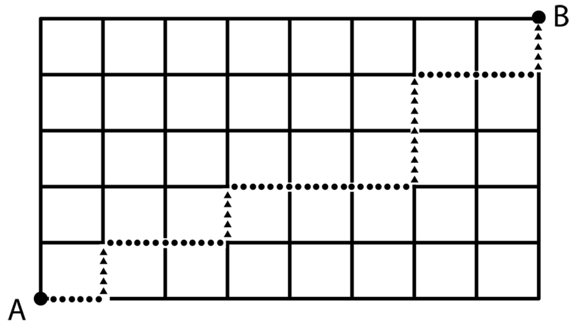

Fig. 1: The path code $\langle i_1, i_2, ..., i_6\rangle = \langle 1,2,3,0,2,0\rangle$.

Now let us go from A to B via a shortest way – their number is the number of coding strings i.e.

$$\binom{n+k-1}{n} = \binom{n+k-1}{k-1}.$$

Of course:

Compositions with the restriction $i_1 \geq i_2 \geq ... \geq i_k \geq a$ are partitions.
Note *that parts of compositions might be zero if $a = 0$.*

Consider now partitions and compositions with pairwise distinct *nonzero parts.*

Definition 1.2 Let us introduce the notation:
$\Pi[n,k]$ = the number of **partitions** of n into k **distinct nonzero parts**.
$C[n,k]$ = the number of **compositionss** of n into k **distinct nonzero parts**.

Exercise 1.2. Prove that $\Pi[0,0] = 1,\ \Pi[n,k] = 0$ if $n < 0$ and

$$\Pi[n,k] = \Pi[n-k,k] + \Pi[n-k,k-1].$$

Solution. Start. Indeed.
$\Pi[n-k,k-1]$ = the number of **partitions** of $n-k$ into $k-1$ distinct non-zero parts = the number of **partitions** of n into k distinct non-zero parts with the **smallest part equal to one**. Indeed – just look at this:

$$5+4+3+2+1 \quad \Leftrightarrow \quad (5-1)+(4-1)+(3-1)+(2-1)\ [+\ zero]$$

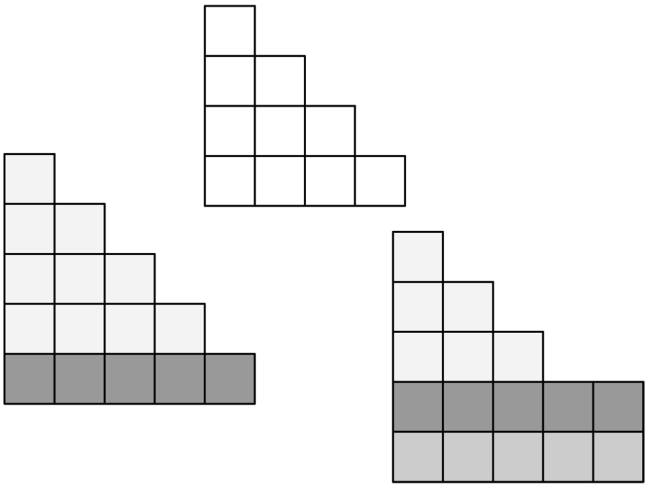

Fig. 2: description

k parts of n with smallest part equal to one $\Leftrightarrow$ $k-1$ parts of $n-k$. Here equivalence $\Leftrightarrow$ is achieved via cutoff of k ones ("unite boxes") – see Fig.2.

Now cut off k unite boxes from k parts with smallest part **greater than one**. Then $\Pi[n-k,k]$ = the number of **partitions** of $n-k$ into k distinct non-zero parts = the number of partitions of n into k distinct non-zero parts with the smallest part **greater than one**. Indeed – just look at this:

$$6+4+3+2 \quad \Leftrightarrow \quad (6-1)+(4-1)+(3-1)+(2-1),$$

so k parts of n with smallest part greater than one $\Leftrightarrow$ k distinct parts of $n-k$.

Exercise 1.3. Print a finite part of the Pascal – like triangle given by the infinite array with $\Pi[n,k]$ as matrix elements. (See: Mathemagics in
http : //ii.uwb.edu.pl/akk/index.html).
Of course:

the number of compositions $C[n,k] = k!$ number of partitions $\Pi[n,k]$.

Of course (Why *"Of course"*?) $C[0,0] = 1$, $C[n,k] = 0$ if $n < 0$,

$$C[n,k] = C[n-k,k] + kC[n-k,k-1].$$

Why "Of course"? Because there are $\uparrow k$ places where from might be found one box to be cut off.
Note: $C[n-k,k-1]$ = the number of **compositions** of n into k distinct non-zero parts with the smallest part **equal to one**.
$C[n-k,k]$ = the number of **compositions** of n into k distinct non-zero parts with the smallest part **greater than one**.

Exercise 1.4. Print a finite part of the Pascal-like triangle given by the infinite array with $C[n,k]$ as matrix elements.

Compositions of n into k distinct parts problem I.

The number of compositions of n into k distinct parts = ?

Find any compact formula for the answer.

Any compact formula as for example we have for Stirling numbers, Newton binomial or Gauss q-binomial numbers etc. (See: Mathemagics in *http://ii.uwb.edu.pl/akk/*).

2. Graph composition i.e. "comppartition" of a graph – Basic enumerations

In this section proofs and hints follow [2]. In order to establish notation let us recall the meaning of the basic notions to be used in what follows. We shall consider finite, undirected, labeled graphs, with no loops or multiple edges. The edge with endpoints v_1 and v_2 is $\langle v_1, v_2\rangle$. The set of vertices of graph G is denoted by and the set of edges is denoted by $E(G)$. The notion we need now is that of an **induced** subgraph of the given graph G.

A subgraph H of a graph G is said to be induced if, for any pair of vertices x and y of H, $\langle v_1, v_2\rangle$ is an edge of H if and only if $\langle v_1, v_2\rangle$ is an edge of G. In other words, H is an **induced** subgraph of G if it has the most edges from $E(G)$ over the same vertex set $V(H)$. The idea of specific vertex partition of a graph - named by A. Knopfmacher and M. E. Mays [2] – the graph composition, generalizes both ordinary compositions of positive integers and partitions of finite sets. As for the name we have experienced that it might be nowadays somewhat misleading therefore let us make aware of two notifications right after the Definition 2.1. and related observation.

Definition 2.1 ("comppartition" = composition in [2]) Let $G = \langle E(G), V(G)\rangle$ denotes a labelled graph. A **comppartition** of G is a **partition** of the vertex set $V(G)$ into vertex subsets of the **connected induced** *subgraphs* of G i.e. such a vertex comp-partition of G provides a **set** of **connected included** subgraphs of $G \to G_1, G_2, ..., G_m, G_i = \langle E(G_i), V(G_i)\rangle, i = 1, ..., m$.

Observation: It is important that $G_i = \langle E(G_i), V(G_i)\rangle$ are **induced** subgraphs because the **same** vertex subset may be spanned by different **edge** subsets - therefore to the same **comppartition** of G which is a partition of vertex set $V(G)$ there might correspond different families $\{G_1, G_2, ..., G_m\} \neq \{G'_1, G'_2, ..., G'_m\}$ of connected subgraphs.

> $K_{n,m}$ complete biparite graph has $n \cdot m$ edges linking the first row n dots in such a way that each dot of this row is linked to every one of the second row of m dots (dots = vertices).

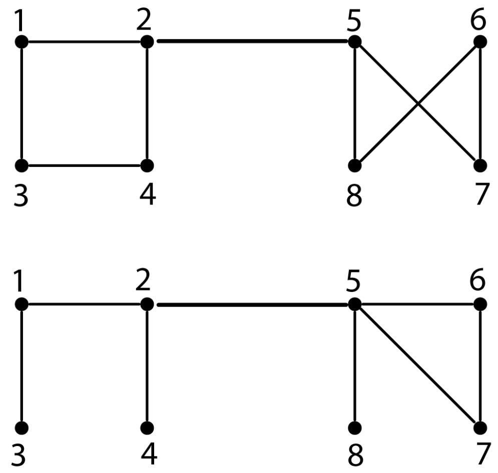

Fig. 3: description

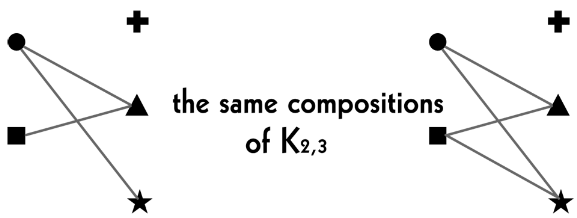

Fig. 4: Bipartite complete graph $K_{2,3}$ compositions‘ example

1. Note that the graph composition-partition introduced by A. Knopfmacher and M. E. Mays is not the textbook composition defined as follows. The composition $G = G_1[G_2]$ of graphs G_1 and G_2 with disjoint point sets V_1 and V_2 and edge sets X_1 and X_2 is the graph with point vertex $V = V1 \times V2$ and such that $u = (u_1, u_2)$ adjacent with $v = (v_1, v_2)$ whenever $[u_1 \text{ adj } v_1]$ or $[u_1 = v_1 \text{ and } u_2 \text{ adj } v_2]$. It is also called the graph lexicographic product.

2. Note that $2^8 = 256$ different types of graph products may be defined of any given two graphs G and H is a new graph whose vertex set is $V(G) \times V(H)$ and where, for any two vertices (g, h) and (g', h') in the product, the adjacency of those two vertices is determined entirely by the adjacency (or equality, or non-adjacency) of g and g', and that of h and h'.
[*http://mathworld.wolfram.com/GraphComposition.html*] Neither of them is “graph composition” from [2] of course.

Notation: $C(G)$ = the number of distinct compositions of a graph G.

Why the name: *composition* of G in [2] ?

$\downarrow$

Exercise 2.1. Let P_n be the path with n vertices. Prove that $C(P_n) = 2^{n-1}, n > 0, C(P_0) \equiv 1$.

Why not the name: *partition* of G in [2] ?

$\downarrow$

Exercise 2.2. Let K_n be the complete graph on n vertices. Prove that $C(K_n) = B_n, n > 0, C(K_0) \equiv 1$. B_n = Bell numbers = exponential numbers.

Note again: the same **partition** of G which is a partition of vertex set $V(G)$ - may give rise to different families $\{G_1, G_2, ..., G_m\} \neq \{G\ _1, G\ _2, ..., G\ _m\}$ of connected subgraphs.

Why not the name: *comppartition* of G in what follows ?

$\downarrow$

Comment. These two above examples are two extreme cases:
no connected graph G with n vertices can have fewer than $C(P_n)$ and no connected graph G with n vertices can have more than $C(K_n)$ **comppartitions**. i.e. $2^{n-1} \leq C(F_n) \leq B_n$, where F_n is any connected graph with n vertices.

Exercise 2.3. Let $G = G_1 \cup G_2$ and there are no edges from vertices of G_1 to vertices of G_2. Then $C(G) = C(G_1)C(G_2)$. The same holds for G_1 and G_2 having exactly one vertex in common. Prove or rather see this.

Note: one obtains *comppartitions* of G by pairing *comppartitions* of G_1 and G_2 in all possible ways.

Exercise 2.4. Let $G = G_1 \cup G_2$ and there is exactly one edge from a vertex of G_1 to a vertex of G_2 whose removal disconnects G. Then $C(G) = 2C(G1)C(G2)$. Prove or rather see this.
Note: Let e be the distinguished edge between vertices v_j and v_j. Any composition of G can be obtained in exactly two ways: either e is included to supply the component v_i in G_1 and the component v_j in G_2 or not. Thus the count from Exercise 2.3. is now doubled.

Exercise 2.5. Let T_n be any tree with n vertices. Prove that $C(T_n) = 2^{n-1}, n > 0$.

Proof: Use induction. Consider T_{n+1}. Remove one edge. This disconnects T_{n+1} into two parts for which the formula holds. By the result of Exercise 2.4. the proof is accomplished.

Exercise 2.6. Let K_n^- be the complete graph on n vertices with *one* edge removed. Prove that $C(K_n^-) = B_n - B_{n-2}, n > 1$.

Proof: Let e be the *deleted* edge between vertices v_j and v_j. Its deletion affects a composition counted by $C(K_n) = B_n$ only when *the component* containing vertices v_j and v_j consists of *exactly these two* vertices v_j and v_j. Otherwise there is a by-pass in K_n connecting these two vertices v_j and v_j. Therefore from the number of composition counted by $C(K_n) = B_n$ one must substract those compositions for which one of the partition component is v_j, v_j. This restriction rules out exactly $C(K_n - 2) = B_{n-2}$ compositions of K_n.

Exercise 2.7. Let C_n be the cycle graph with n vertices. Prove that $C(C_n) = 2^n - n, n > 0$.

Proof: Delete any edge. The resulting graph becomes P_n with $C(P_n) = 2^{n-1}$. Any composition of P_n is also a composition of C_n. The deleted edge may be reinserted, providing a new composition of C_n - previously not counted *unless* the composition of P_n had been obtained by deleting from P_n either no edge or exactly one edge.

In these cases, reinserting the original deleted edge gives the same composition of C_n: namely the composition consisting of all n vertices. Therefore the total number of distinct compositions of C_n is equal to $C(C_n) = 22^{n-1} - n = 2^n - n$.

Exercise 2.8. (Theorem 9 in [3]) Let L_n be the ladder cycle graph with $2n$ vertices and $3n - 2$ edges as L_n is the product of a path of lengt n and a path of length 2.Prove that $C(L_n)$ does satisfy the following recurrence.

$$(L_1) = 2, C(L_2) = 12, C(Ln) = 6C(Ln - 1) + C(Ln - 2)\ for\ n > 2.$$

Proof: see: Theorem 9 in [3].

Exercise 2.9. Number of Ladders with n rungs “problem”.
$C(L_n) =$? Find the Binet-like formula for the answer. [Jacques Binet (1786-1856)].
Contact: $http://mathworld.wolfram.com/LadderGraph.html$.

3. Number compositions with parts constrained by the leading summand

In this section we follow [3]. Note: here now **again**:
Composition of a natural number is an ordered partition of a natural number.
Ordered partition of a natural number is a constrained composition of a natural number.
(...constrained? yes ! constrained by... answer this question)

Of course: Compositions constrained by the requirement $i_1 \geq i_2 \geq ... \geq i_k \geq a$ are partitions.
Note *that parts of compositions might be zero if* $a = 0$ [1].

Exercise 3.1. Consider the number $f_n(k)$ of compositions of a natural number n into k parts with the strictly largest part in the first position i.e.

$$(>) \qquad i_1 + i_2 + ... + i_k = n, \quad 1 \leq i_1, i_2, ..., i_k, \quad i_1 > i_k \ for \ k > 1$$

Observe-show that the following formula gives the ordinary generating function for these numbers with k fixed :

$$F_k(z) = \sum_{n \geq 0} f_n(k) z^n = \frac{(1-z)z^k}{1 - 2z + z^k}$$

Proof:

$$k \geq 2 \quad \sum_{n \geq 0} f_n(k) z^n = z^k \sum_{n \geq 1} \Phi_n^{k-1} z^n = \frac{z^k}{1 - z - z^2 - ... - z^{k-1}},$$

where higher order Fibonacci sequences are defined

$$\Phi_{n+k-1}^{(k-1)} = \sum_{i=0}^{k-2} \Phi_{n+i}^{(k-1)} ...$$

(... what are the initial values?)

Exercise 3.2. Consider the number $f_n^*(k)$ of compositions of a natural number n into k parts with the largest part in the first position i.e.

$$(\geq) \qquad i_1 + i_2 + ... + i_k = n, \quad 1 \leq i_1, i_2, ..., i_k, \quad i_1 \geq i_k \ for \ k > 1$$

Observe-show that the following formula gives the ordinary generating function for these numbers with k fixed:

$$F_k^*(z) = \sum_{n \geq 0} f_n^*(k) z^n = \frac{(1-z)z^k}{1 - 2z + z^{k+1}}$$

Proof:

$$k \geq 2 \quad \sum_{n \geq 0} f_n^*(k) z^n = z^k \sum_{n \geq 1} \Phi_n^k z^n = \frac{z^k}{1 - z - z^2 - ... - z^k},$$

where higher order Fibonacci sequences are defined

$$\Phi_{n+k}^{(k)} = \sum_{i=0}^{k-1} \Phi_{n+i}^{(k)} ...$$

(... what are the initial values?)

Exercise 3.3. Consider the two following sequences of numbers $\langle f_n \rangle_{n \geq 0}$ and $\langle f_n^* \rangle_{n \geq 0}$ and their ordinary generating functions

$$F(z) = \sum_{n \geq 0} F_n(z) = \sum_{n \geq 0} f_n z^n \quad and \quad F(z) = \sum_{n \geq 0} F_n^*(z) = \sum_{n \geq 0} f_n^* z^n$$

Observe-show that for $n \geq 2$ $f_{n+1} = f_n^*$ *as* $zF^*(z) = F(z) - z$, *(...what are the initial values?)*

Naturally

$f_n = \sum_{k\geq 0} f_n(k)$ = the number of all compositions of a natural number n with the strictly largest part in the first position.

Compositions of n with strictly largest part in the first position **problem II.**
$f_n =$? Find any compact formula for the answer.
A possible way to solve the problem II is to solve the recurrence equation from the next exercise.

Exercise 3.4.
Observe-show that

$$f_n(k) = 2f_{n-1}(k) - f_{n-k}(k) + \delta_{n,k} - \delta_{n,k+1},$$

(...what are the initial values?)
Hint: use

$$F_k(z) = \sum_{n\geq 0} f_n(k) z^n = \frac{(1-z)z^k}{1-2z+z^k}.$$

Compositions of n into k parts with strictly largest part in the first position **problem III**.
$f_n(k) = ?$ Find any compact formula for the answer.

4. Compositions with distinguished part

In this section we follow [4].

Exercise 4.1. Let $C_k(n)$ be the number of compositions of n in which at least one k occurs. Prove then that

$$\sum_{n\geq 0} C_k(n) z^n = \frac{z}{1-2z} - \frac{z - z^k + z^{k+1}}{1-2z+z^k - z^{k+1}}, \quad n \geq k \geq 1.$$

Proof: Consider $C^*k(n)$ to be the number of compositions of n in which no part equals to k. Consider $C_k^*(n,m)$ to be the number of compositions of n into m parts with no part equal to k.

Note: by the product law of generating functions

$$\sum_{n\geq 0} C_k^*(n,m) z^n = (z + z^2 + ... + z^{k-1} + z^{k+1} + ...)^m$$

and **sum** now the above over m via $\sum_{m=1}^{n}$ and then

Observe that

$$\frac{z - z^k + z^{k+1}}{1-2z+z^k-z^{k+1}} \equiv \frac{q}{1-q}. \qquad q =?$$

That is all ... as of course

$$\sum_{n\geq 0} C(n)z^n = ... = \frac{z}{1-2z}.$$

... and $q =$? Answer:

$$q = \frac{z - z^k + z^{k+1}}{1-z}.$$

Exercise 4.2. $C^*k(n)$ to be the number of compositions of n in which no part equals to k.
Then

$$C_k^*(n) = 2C_k^*(n-1) - C_k^*(n-k) + C_k^*(n-k+1), \quad n \geq k+2$$

(... *what are the initial values*?)

Proof: see [4] for two proofs of the above. One of them is combinatorial.

Compositions of n with no $k \in \mathbb{N}$ summand **problem IV**.

$C_k^*(n) =$? Find any compact formula for the answer.

5. Number compositions with all part distinct

For this section see [5]. See also related [6]. We come back to the very first question posed at the start of this article.

"The number of compositions of n into k **all** parts **distinct = ?** *... that is the question...*" Recalling now the Definition 2.1. and using the result of Exercise 2.1. we proved what follows.

Recall 5.1. Prove that $C[0,0] = 1$, $C[n,k] = 0$ if $n < 0$ and

$$C[n,k] = C[n-k,k] + kC[n-k,k-1].$$

Let us now introduce also the overall number of compositions into distinct parts.

Definition 5.1. $C[n] = \sum_{k\geq 1} C[n,k]$ *= the number of all compositions of n into* ***distinct*** *parts.*

Information from [5]: one may prove that

$$C[z] = \sum_{n\geq 1} C[n] = \sum_{k\geq 1} \frac{k! z^{\binom{k+1}{2}}}{(1-z)(1-z^2)...(1-z^k)}$$

So what about our question now?

Compositions of n into **distinct** parts **problem V**.

The number of compositions of n into **distinct** parts i.e. $C[n] =$?

Find any compact formula for the answer.

To this end and for any case we add also the miscellaneous Appendix on formulas with intrinsic reference to unavoidable numbers' compositions therein.

6. Appendix "Disce puer"

Consider Stirling numbers [Knuth notation] of the second $\left\{ {n \atop k} \right\}$ and of the first kind $\left[{n \atop k} \right]$ via compositions of the natural number n **summation**. (See: Mathemagics in *http://ii.uwb.edu.pl/akk/*).

Exercise 1.A. Recall [7,8,9], prove adn compare

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{\substack{i_1+i_2+...+i_k=n \\ 0<i_1,...,i_k\leq n}} \binom{n}{i_1, i_2, ..., i_k}$$

where

$$\binom{n}{i_1, i_2, ..., i_k} = \frac{n!}{i_1! i_2! ... i_k!}$$

and

$$(\mathbf{D}^*) \qquad i_1 + i_2 + ... + i_k = n, \quad 0 < i_1, i_2, ..., i_k \leq n$$

thereby $\langle i_1, i_2, ..., i_k \rangle$ is a vector solution of (D^*) i.e. a composition of the natural number n.

Now compare

$$\left\{ {n \atop k} \right\} = \frac{n!}{k!} \sum_{\substack{i_1+i_2+...+i_k=n \\ 0<i_1,...,i_k\leq n}} \frac{1}{i_1! i_2! ... i_k!}$$

with [8]

$$\left[{n \atop k} \right] = \frac{n!}{k!} \sum_{\substack{i_1+i_2+...+i_k=n \\ 0<i_1,...,i_k\leq n}} \frac{1}{i_1 i_2 ... i_k}$$

Exercise 2.A. **Ad Cobweb tiling problem** [10] see also [11,12]. $\left\{ {\eta \atop \kappa} \right\}_\lambda$ =? i.e. the number of κ-block partitions with block sizes all equal to λ = ? The answer is known due to [7,8,9,10]. Prove it.

$$\left\{ {\eta \atop \kappa} \right\}_\lambda = \delta_{\eta,\kappa\lambda} \frac{\eta!}{\kappa!} \sum_{\substack{i_1+i_2+...+i_\kappa=\eta \\ 0<i_1=...=i_\kappa=\lambda}} \frac{1}{i_1 i_2 ... i_\kappa} = \delta_{\eta,\kappa\lambda} \frac{\eta!}{\kappa!(\lambda!)^\kappa}$$

Hint to Exercise 1A.

$$|X^S| = |X|^|S| = k^n$$

therefore (k summands)

$$k^n = (1 + 1 + ... + 1)n = \sum_{i_1+i_2+...+i_k=n} \binom{n}{i_1, i_2, ..., i_k}$$

where $0 < i_1, i_2, ..., i_k = n$. A map from S to X is not a surjection iff $i_1 = 0 \vee i_2 = 0 \vee ... \vee i_k = 0$.

Consider Compositions of the natural number n via **partitions summation**

Exercise 3.A. Prove that

$$C(n, k, 1, \infty) = \binom{n-1}{k-1} = \sum_{\substack{1\lambda_1+2\lambda_2+...+n\lambda_n=n \\ \lambda_1+\lambda_2+...+\lambda_n=k}} \frac{k!}{\lambda_1!\lambda_2!...\lambda_n!}$$

7. Appendix : "what is further on"

7.1. Recall. The idea of graph compositions as introduced by A. Knopfmacher and M. E. Mays [2], generalizes both ordinary compositions of positive integers and partitions of finite sets. In [2–5] the authors provided various formulas, generating functions, and recurrence relations for composition counting functions for several families of graphs. Recently in [13] some of the results involving compositions of bipartite graphs have been derived in a simpler way using exponential generating functions.

7.2. In the [14] a new construction of tree-like graphs where nodes are graphs themselves was added and examples of these tree-like compositions, a corresponding theorem and few resulting conclusions are delivered. Compare with [15].

7.3. Ad naming and history roots see and compare with [2] the content of references [16–19]. This comparison is important. For example, in [16] W.H. Cunningham described a composition for directed graphs, a composition which generalizes the substitution (or X-join) composition of graphs and digraphs, as well as the graph version of set-family composition. There he s proved that a general decomposition theory from [17] can be applied to the resulting digraph decomposition. "A consequence is a theorem which asserts the uniqueness of a decomposition of any digraph, each member of the decomposition being either indecomposable or "special". The special digraphs are completely characterized; they are members of a few interesting classes. Efficient decomposition algorithms are also presented" quoted from [16]. As for the Cunningham's split decomposition of an arbitrary undirected graph – the related composition operation is a generalization of the modular composition – also called substitution of X-join. The split decomposition is useful in recognizing special classes of graphs, such as circle graphs, which are the intersection graphs of arcs of a circle, and parity graphs, because these graphs are closed under the inverse composition operation. The decomposition can also be used to find NC algorithms for some optimization problems on special families of graphs, assuming these problems can be solved in NC for the indecomposable graphs of the decomposition.[see more in *http://hdl.handle.net/1813/6874*] Let us illustrate the importance of Cunningham‘s split decomposition from [16] quoting as an example the reference [20]. The authors of [20] introduce there a new structural property of parity graphs. Namely Cunningham‘s split decomposition returns exactly, as building blocks of parity graphs, cliques and bipartite graphs. This characterization and the observation that the split decomposition process is performed in linear time, gives rise to optimum algorithms for the recognition problem and for the maximum weighted clique problem.

References

[1] C. Kimberling, *Enumeration of paths, compositions of integers, and Fibonacci numbers*, The Fibonacci Quarterly, 39 (5), (2001), 430–435.

[2] A. Knopfmacher and M. E. Mays, *Graph compositions I: Basic enumeration Integers*, The Electronic Journal of Combinatorial Number Theory **1** (2001), A04.

[3] A. Knopfmacher and N. Robbins, *Compositions with parts constrained by the leading summand*, Ars Combinatoria **76** (2005), 287-295.

[4] A. Knopfmacher and M. E. Mays, *Compositions with m distinct parts*, Ars Combinatoria **53** (1999) 111–128.

[5] B. Richmont and A. Knopfmacher, *Compositions with distinct parts*, Aequationes Math. **49** (1995), 86–97.

[6] T. Doslic, *Maximum Product Over Partitions Into Distinct Parts*, Journal of Integer Sequences **8** (2005), Article 05.5.8.

[7] Ch. Jordan, *On Stirling Numbers*, Thoku Math. J. **37** (1933), 254–278, formula (59).

[8] C. Jordan, *Calculus of finite differences*, Chelsea, New York, 1947, 51, 60.

[9] T. Kreid, *Combinatorial sequences of polynomials*, Comment. Math. Prace Math. **29** (1990), 233–242.

[10] A. K. Kwaśniewski, *Cobweb posets as noncommutative prefabs*, Adv. Stud. Contemp. Math. **14** (1) (2007), 37–47.

[11] A. K. Kwaśniewski, *On cobweb posets and their combinatorially admissible sequences*, Adv. Studies Contemp. Math. Vol. **18**, no. 1 (2009), 17–32; ArXiv:0512578v4 [v5] Mon, 19 Jan 2009 21:47:32 GMT.

[12] M. Dziemiańczuk, *On Cobweb posets tiling problem*, arXiv: math.CO/0709.4263 4 Oct 2007.

[13] A. Huq, *Compositions of Graphs Revisited*, The Electronic Journal of Combinatorics **14** (2007), N15; arXiv:math.CO/0704.3821, 28 Apr 2007.

[14] W. Bajguz, *Graph and Union of Graphs Compositions*, Advanced Studies in Contemporary Math. Vol. 16, No. 2, April 2008, pp. 245-249. arXiv:math.CO/0601755, 31 Jan 2006.

[15] J. N. Ridley and M. E. Mays, *Compositions of Union of Graphs*, The Fibonacci Quarterly **42**, no. 3 (2004), 222–230.

[16] W. H. Cunningham, *Decomposition of Directed Graphs*, SIAM J. Alg. Disc. Meth. **3** (1982), 214–228.

[17] W. H. Cunningham and J. Edmonds, *A Combinatorial Decomposition Theory*, Canad. J. Math. **32** (1980), 734–765.

[18] L. O. James, R. G. Stanton, and D. D. Cowan, *Graph decomposition for undirected graphs*, Proceedings of the third Southeastern International Conference on Combinatorics, Graph Theory, and Computing (CGTC'72) (1972), 281–290.

[19] G. Sabidussi, *The composition of graphs*, Duke Math. J. **26** (1959), 693–696.
[20] S. Cicerone and G. Di Stefano, *On the extension of bipartite to parity graphs*, Discrete Applied Mathematics **95**, Issues 1–3, (1999), 181–195.

Institute of Combinatorics and its Applications
High School of Mathematics and Applied Informatics
Kamienna 17, PL-15-021 Białystok
Poland
e-mail: kwandr@gmail.com

Presented by Marek Moneta at the Session of the Mathematical-Physical Commission of the Łódź Society of Sciences and Arts on November 4, 2009

O SKŁADANIU LICZB I GRAFÓW

S t r e s z c z e n i e

Głównym celem obecnej noty jest zaproponowanie kilku problemów, które dają się łatwo sformułować, lecz niektóre z nich zapewne nie są rozwiązane. Są one o ogólnym znaczeniu dla matematyki dyskretnej włączając nową gałązkę w ramach pewnej gałęzi teorii grafów związanej z ich składaniem. Wynikają stąd problemy, które są istotne w związku z szeregiem zadań i wskazówek opartych przede wszystkim na ostatnich pracach Knopfmachera i współautorów.