ON HYPERBOLIC AND ELLIPTIC MAPPINGS AND QUASI-NUMBER ALGEBRAS

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ABSTRACT:

Generalizations of complex numbers suggested by Weierstrass, Bruwier, Mikusiński et. al. are investigated in detail. These generalizations are intrinsicly related to hyperbolic and elliptic mappings via polar representation of "quasi-numbers" proposed by Fleury et. al.

Relevance of these quasinumber systems to Potts-like models, fractals and Weyl's finite quantum mechanics with discrete configuration space is indicated.

Introduction:

Hyperbolic mappings of matrix arguments arise naturally in applications ranging from statistical physics to quantum physics [1-5]. They appear to be natural maps everywhere there where generalized Clifford algebras play a role [2,3] and/or - there, where k-ubic forms instead of quadratic ones serve to introduce additional structures (see [3]).

These diverse applications amplify the need to study more systematically the number-like structure of hyperbolons and elliptons as we call - in short - elements of quasi-composition algebras that emerge in the sequel as natural generalizations of complex numbers.

The hyperbolon & ellipton algebras appear to be special abelian subalgebras of generalized Clifford algebras [6,7].

This paper is inspired by the recent progress in the study of such algebras made by the authors of [8] who rediscovered these algebras in a very fruitful way.

Here we present a history in brief of the subject, not pretending naturally, to make the complete list of references.

As noticed by the authors of [8] the ideas leading to complex numbers seem to stem from the ancient Greece [9].

The complex number system might be viewed on as the commutative two dimensional composition algebra over $\mathbb{R}$ and it

is isomorphic to the real, division, commutative *subalgebra of* **anti***circulant* matrices of $M_2(\mathbb{R})$ i.e.

$$\mathbb{C} \simeq A_-(2;\mathbb{R}) \equiv \left\{ \begin{bmatrix} x & -y \\ y & x \end{bmatrix}, \quad x,y \in \mathbb{R} \right\},$$

with composition norm given by $\|z\|^2 \equiv \det z \;; \quad z \in A_-(2;\mathbb{R})$.

[Note that parallely one has commutative , two dimensional "quasi-composition" algebra over $\mathbb{R}$ which is isomorphic to the real, **NOT**division, commutative subalgebra of circulant matrices of $M_2(\mathbb{R})$ i.e.

$$A_+(2;\mathbb{R}) \equiv \left\{ \begin{bmatrix} x & y \\ y & x \end{bmatrix}, \quad x,y \in \mathbb{R} \right\}.$$

with composition "quasi-norm" given by

$$\|h\|^2 \equiv \det h \;; \quad h \in A_+(2;\mathbb{R}).$$]

The known extension of real numbers considered as **division** algebra over reals leads (via Cayley-Dickson procedure and Hurwitz theorem [10]) to complex numbers, quaternions and octonions.

Another extension of $\mathbb{R}$, $\mathbb{C}$ commutative number systems and quaternion $\mathbb{Q}$ non commutative number-like system, viewed upon as **associative** algebras over reals **generated** by elements satisfying appropriate relations, resulted from corresponding quadratic form - leads to $\mathscr{C}(p,q)$ Clifford algebras [11]. (Elements of $\mathscr{C}(p,q)$ Clifford algebras are called by some authors - *"geometric numbers"* or *"Clifford numbers"*).

These were still generalized by Morinaga, Nono et.al. [6,7] to become a case of the generalized Clifford algebras.

(As indicated in [3] the generalized Clifford algebras are related to k-ubic forms similarily as usual Clifford algebras are defined via "2-ubic i.e. quadratic forms. For peculiarities of this assignment - see [3].)

These extensions however are **noncommutative** *with* **composition** *and* **division** *properties* - **lost.**

A possible **commutative extension** of complex numbers to the case of arbitrary number of real dimensions was already considered by **Weierstrass** [12]. The division property is then lost but the composition property survives under the condition that instead of "length" one considers a **volume** of such quasi numbers. (The authors of [8] call the n-th root of the volume of a quasi- number a "*pseudo-norm*"; but this seems to be misleading).

As we shall see, in matrix representation, such quasi-numbers are given by anticircular matrices

$$X = \begin{bmatrix} x_0 & -x_{n-1} & \cdots\cdots & -x_1 \\ x_1 & x_0 & \cdots\cdots & -x_2 \\ \vdots & & & \\ x_{n-1} & x_{n-2} & \cdots\cdots & x_0 \end{bmatrix},$$

and we shall call them **elliptons** for the reasons to be apparent soon.

Parallely one can extend the notion of the "hyperbolic quasi-numbers" $A_+(2;\mathbb{R})$ to the corresponding $A_+(n;\mathbb{C})$ algebra of circulant matrices which we shall call **"hyperbolons"** i.e.

$$Y = \begin{bmatrix} y_0 & y_{n-1} & \cdots\cdots & y_1 \\ y_1 & y_0 & \cdots\cdots & y_2 \\ \vdots & & & \\ y_{n-1} & y_{n-2} & \cdots\cdots & y_0 \end{bmatrix}$$

The interest in such a quasi-number system we owe already to Bruwier (1949) [13] who considered algebras spanned by "hyperbolic units" $1, \omega, \omega^2, \ldots, \omega^{n-1}$, with the multiplication table being specified by

$$\omega^i \omega^j = \omega^{i \dot{+} j} \; ; \; i,j \in Z'_n \; ;$$

where ω *is the primitive n-th root of unity* and Z'_n stands for the cyclic group in additive realization.

In that way He arrives immediately to **hyperbolic function** notion as a hypercomplex number (or "hyperbolon") $\exp(\omega x)$ is the linear combination of those "hyperbolic units" $1, \omega, \omega^2, \ldots, \omega^{n-1}$ with hyperbolic functions $h_i(x)$; $i \in Z'_n$ as coefficients ; namely we have :

$$h_i(x) := 1/n \sum_{k \in Z'_n} \omega^{-ki} \exp(\omega^k x) \quad ; \quad i \in Z'_n \, .$$

$$\exp(\omega^k x) \equiv \sum_{i \in Z'_n} \omega^{ki} h_i(\varphi) \; ; \quad k \in Z'_n \, .$$

(Note that as a matter of fact we are dealing here with harmonic analysis over the finite abelian group Z'_n.)

In complete analogy one can extend the Weierstrass [12] idea of introducing "more" imaginary units, what leads to the analogous formulas for elliptic functions $f_l(x)$; $l \in Z'_n$.

These are formulas typical for "elliptons"; namely

$$f_l(x) = \kappa^{-l}/n \sum_{k \in Z'_n} \omega^{-kl} \exp(\omega^k \kappa x) \quad ; \quad l \in Z'_n \ ;$$

$$\exp\left[(\kappa\omega)^k \varphi\right] = \sum_{l \in Z'_n} \kappa^{(2k+1)l} f_l(\varphi) \quad ; \quad k \in Z'_n$$

where κ *is the primitive 2n-th root of unity.*

The elliptic functions $f_l(x)$; $l \in Z'_n$, as can be easily seen, are equivalently defined by

$$f_l(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{nk+l}}{(nk+l)!}$$

These functions were already studied by Jan G. Mikusiński in [14], which is merely devoted to finding zeros of these elliptic functions. As shall be clear soon - elliptic functions $f_l(x)$; $l \in Z'_n$ of Mikusiński are naturally related to ellиptons via polar decomposition of nonzero-volume quasinumbers.

Hyperbolic and elliptic functions are of great interest on their own and were studied apart from Mikusiński also by Oniga [15], who proved de Moivre formulas for both kinds of these functions i.e. for hyperbolic functions $h_i(x)$; $i \in Z'_n$ and for elliptic functions $f_l(x)$; $l \in Z'_n$. Here we show that de Moivre formulas are immediate consequence of the group property of a certain subset of volume-one quasinumbers.

Hyperbolic and elliptic functions were studied - in more algebraic than analytic manner - also by Y. Lehrer [16] and L. L. Silverman [17]. A detailed bibliography on hyperbolic and elliptic functions studies at the early period of interest in them, can be found in the 1949 L. Poli paper [18].

After that time these functions and related algebraic structures have appeared in numerous applications in mathematical physics as incompletely illustrated by quoted references [5] (see also [4]) Here we show that quasinumbers algebra structure in an inevitable context for hyperbolic and elliptic functions to operate with - easily and in almost full correspondence with formulas known in complex numbers case. Of particular importance in this connection is the fruitful idea of polar representation introduced by the authors of [8] in 1991, for whom it served to rediscover "multisinus" and "multihyperbolic" functions - as they called the hyperbolic and elliptic functions

$$h_i(x), \ f_l(x); \ i, l \in Z'_n$$

In our paper we present a systematic study of hyperbolon and ellipton algebras, which appear to admit a quasi conjugation and which are volume-composition algebras.

The role of hyperbolic and elliptic functions $h_i(x)$, $f_l(x)$; $i,l \in Z'_n$ played in these algebras is apparent from the polar representation of nonzero-volume quasinumbers X (hyperbolon) and Y (ellipton) i.e.

$$X = \prod_{s \in Z'_n} H_s(\varphi_s),$$

$$Y = \prod_{s \in Z'_n} F_s(\varphi_s),$$

where

$$H_1(x) = \begin{bmatrix} h_o(x) & h_{n-1}(x) & \cdots\cdots h_1(x) \\ h_1(x) & h_o(x) & \cdots\cdots h_2(x) \\ \vdots & & \\ h_{n-1}(x) & h_{n-2}(x) & \cdots\cdots h_o(x) \end{bmatrix},$$

$$F_1(x) = \begin{bmatrix} f_o(x) & -f_{n-1}(x) & \cdots\cdots -f_1(x) \\ f_1(x) & f_0(x) & \cdots\cdots -f_2(x) \\ \vdots & & \\ f_{n-1}(x) & f_{n-2}(x) & \cdots\cdots f_o(x) \end{bmatrix},$$

and $H_s(\varphi)$ & $F_s(\varphi)$; $s = 2,\ldots,n-1$ are of similar type (defined later). For $n = 2$ and $x \in \mathbb{C}$ we then get

$$F_1(x) = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix},$$

$$H_1(x) = \begin{bmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{bmatrix},$$

while $H_o(\varphi_0) = \mathbb{I}\sqrt[n]{\det X} \neq 0$ and $F_o(\varphi_0) = \mathbb{I}\sqrt[n]{\det Y} \neq 0$.

An immediate application of hyperbolons to physics relies on the observation that transfer matrix in Potts model is a product of a hyperbolon and a dual hyperbolon; the latter to be defined later.

A quite fascinating relevance of hyperbolon algebra and its dual to the Weyl's quantum mechanics on cyclic group Z_n is obtained via identification of unitary transformations in Weyl's commutation relations

$$VU = \exp\{2\pi i/n\}UV,$$

with corresponding hyperbolon quasinumbers V & U.

As a matter of fact, it seems to be Herman Weyl then, who was the first discoverer (1932) of generalized Pauli algebra [24] (see also [20] & [19]). The Heisenberg group then is identical with generalized Dirac or Pauli group - in Clifford algebra language [3].

II. MATRIX REPRESENTATION OF ELLIPTIC AND HYPERBOLIC QUASINUMBERS

Consider any element $h \in A_+$ i.e.

$$h = \sum_{l \in Z'_n} h_l U^l$$

The element $h \in A_+$ is represented via left regular representation of A_+ algebra by a circulant matrix H

$$H = \begin{bmatrix} h_0 & h_{n-1} & \cdots & h_1 \\ h_1 & h_0 & \cdots & h_2 \\ \vdots & & & \vdots \\ h_{n-1} & h_{n-2} & \cdots & h_0 \end{bmatrix},$$

as readily seen from the definition of representing matrix

$$\sum_{j \in Z'_n} h_{jl} U^j := h U^l = \sum_{k \in Z'_n} h_k U^{k \dot{+} l} = \sum_{j \in Z'_n} h_{j \dot{-} l} U^j$$

(Note: $\dot{+}$ & $\dot{-}$ mean addition and subtraction in Z'_n ring.)

Similarly any element $f \in A_-$ i.e.

$$f = \sum_{l \in Z'_n} f_l J^l$$

is represented via left regular representation of A_- algebra by an anticirculant matrix F

$$F = \begin{bmatrix} f_0 & -f_{n-1} & \cdots & -f_1 \\ f_1 & f_0 & \cdots & -f_2 \\ \vdots & & & \vdots \\ f_{n-1} & f_{n-2} & \cdots & f_0 \end{bmatrix},$$

as readily seen from the definition of representing matrix

$$f\,U^l = \sum_{j \in Z'_n} f_{jl} U^j = \sum_{k \in Z'_n} f_k U^{k \dot{+} l} = \sum_{j \in Z'_n} \alpha(j \dot{-} l) f_{j \dot{-} l} U^j$$

Because of

$$f = \sum_{l \in Z'_n} f_l J^l,$$

we have

$$F = \sum_{l \in Z'_n} f_l \gamma_-^l,$$

where

$$\gamma_- = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 \end{bmatrix}.$$

The matrix γ_- is just left-regular representative of an endomorphism J, and of course $\gamma_-^n = -\mathbb{I}$. (The corresponding matrix for hyperbolic case is

$$\gamma_+ = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & & & & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{bmatrix},$$

being in this case just left-regular representative of an endomorphism U.)

The eigenvalues of γ_- are given by the following set

$$\left\{ \exp\left[\frac{i\pi(2k+1)}{n} \right] \right\}_{k \in Z'_n},$$

as follows from $\gamma_-^n = -\mathbb{I}$.

The eigenvalues of γ_+ are given correspondingly by

$$\left\{ \exp\left[\frac{i2\pi k}{n} \right] \right\}_{k \in Z'_n},$$

due to $\gamma_+^n = \mathbb{I}$.

Hence eigenvalues of F anticirculant matrix are given by

$$\left\{ \lambda_k = \sum_{s \in Z'_n} f_s \left[\exp\left[\frac{i\pi(2k+1)}{n} \right] \right]^s \right\}_{k \in Z'_n}.$$

Correspondingly the eigenvalues of H circulant matrix are given by

$$\left\{ \kappa_k = \sum_{s \in Z'_n} h_s \exp\left[\frac{i2\pi k}{n} s \right] \right\}_{k \in Z'_n}.$$

III. GENERALIZED HYPERBOLIC AND ELIPTIC MAPPINGS

In this short section we just outline an immediate generalization of ..ic and hyperbolic functions as this generalization is of big importance for the use of transfer matrix technique in Potts-like models.

Let $\omega \equiv \exp(2\pi i/n)$ be the primitive n-th root of unity.

Let $x \in M_N(\mathbb{C})$. Let $Z'_n = \left\{ 0,1,\dots,n-1 \right\}$ denotes the module n additive group. Then we might define mappings

Definition 1. $h_i : M_N(\mathbb{C}) \dashrightarrow M_N(\mathbb{C})$; $i \in Z'_n$;

$$h_i(\cdot) = \sum_{k=0}^{\infty} \frac{x^{nk+i}}{(nk+i)!}.$$

Naming: $\left\{ h_i \right\}_{i \in Z'}$ shall be called *hyperbolic mappings*.

Right from the definition we have:

$$(1) \qquad \sum_{i=0}^{n-1} h_i(x) = e^x ;$$

$$(2) \qquad h_i(\omega x) = \omega^i h_i(x) \quad ; \; i \in Z'_n ;$$

From (1) & (2) one easily derives the "Euleur formulas" [2]

$$(3) \qquad h_i(x) = 1/n \sum_{k \in Z'_n} \omega^{-ki} \exp(\omega^k x) \quad ; \; i \in Z'_n ,$$

which are, as a matter of fact harmonic analysis over Z'_n group formulas, the inverse transformation being given by

$$\exp(\omega^k x) = \sum_{l \in Z'_n} \omega^{kl} h_l(x) \ ; \ k \in Z'_n .$$

For $x \in \mathbb{R}$ and $n = 2$ one recovers

$$\cosh \equiv h_0 \quad \& \quad \sinh \equiv h_1$$

functions.

Most of the identities satisfied by cosh & sinh functions generalize straightforwardly to the case of general hyperbolic mappings as for example ([2], [16], [17]):

$$(*) \qquad \sum_{l \in Z'_n} h_l(x) h_{k \dot{-} l}(y) = h_k(x \dot{+} y) \ ; \ l, k \in Z'_n .$$

(Note: $\dot{+}$ & $\dot{-}$ mean addition and subtraction in Z'_n ring.)

Definition 2. $f_l : M_N(\mathbb{C}) \dashrightarrow M_N(\mathbb{C})$; $l \in Z'_n$;

$$f_l(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{nk+l}}{(nk + l)!} .$$

Naming: $\left\{ f_l \right\}_{l \in Z'_n}$ shall be called *elliptic mappings.*

Right from the definition we have:

$$(1') \qquad \sum_{l=0}^{n-1} \kappa^l f_l(x) = e^{\kappa x};$$

where

$$\kappa \equiv \exp\{ i\pi/n \}.$$

$$(2') \qquad f_l(\omega x) = \omega^l f_l(x) \qquad ; \ l \in Z'_n;$$

From (1') & (2') one easily derives the "Euleur formulas":

$$(3') \qquad f_l(x) = \kappa^{-l}/n \sum_{k=0}^{n-1} \omega^{-kl} \exp(\omega^k \kappa x) \quad ; \quad l \in Z'_n .$$

Note also that

$$\kappa^l f_l(x) \equiv h_l(\kappa x), \ l \in Z'_n .$$

For $x \in \mathbb{R}$ and $n = 2$ one recovers

$$\cos \equiv f_0 \quad \& \quad \sin \equiv f_1$$

functions.

Most of the identities satisfied by cos & sin functions generalize straightforwardly to the case of general elliptic mappings as for example

$$\sum_{l=0}^{n-1} f_l(x) f_{k \dot{-} l}(y) = f_k(x \dot{+} y) \ ; \ l, k \in Z'_n .$$

IV. QUASI-CONJUGATION OF QUASINUMBERS

OBSERVATION A: Let $F \in A_{\mp}$ & $\det F \neq 0$; then $F^{-1} = \dfrac{F^{*}}{\det F}$;

where F^{*} denotes the matrix adjoint to F i.e.

$$F^{*}_{ij} := (-1)^{i+j} M_{ji} ,$$

M_{ij} being the ij-th minor of F matrix.

The $*$ - mapping ; $* : M_n(\mathbb{C}) \dashrightarrow M_n(\mathbb{C})$ has the following obvious properties:

1) $(A^{*})^{*} \equiv A\,(\det A)^{n-2}$ &

2) $\boxed{(AB)^{*} \equiv B^{*}A^{*}}$ $\forall\ A, B \in Gl(n;\mathbb{C})$;

and for *any matrix* including cyclic or anticyclic matrices

3) $\det A \det A^{*} = (\det A)^{n} \iff AA^{*} \equiv \mathbb{I} \det A$;

4) $\det A^{*} = (\det A)^{n-1}$

5) $\det A = 0$ iff $\det A^{*} = 0$.

(**Remark:** $(A^{*})^{*} = A$; $A \in M_n(\mathbb{C})$; $(\det A)^{n-2} = 1$; therefore for generators of generalized Clifford algebra (including matrices $\gamma_1 \equiv \gamma_+$ & $\gamma_3 = 1/n\ C^{-1}\gamma_1 C$; see sect. IX) we have $(\gamma^{*})^{*} = \gamma$.)

Thus we see that the $*$ - mapping: $* : M_n(\mathbb{C}) \dashrightarrow M_n(\mathbb{C})$ becomes linear iff $n = 2$ and in this case , because of 1) it becomes also an involution. For commutative algebras of cyclic and anticyclic subalgebras i.e. for $A_{\pm}(2;\mathbb{C}) \subset M_2(\mathbb{C})$ it is the well known conjugation:

$$A_{+}(2;\mathbb{C}) \ni \begin{bmatrix} x & y \\ y & x \end{bmatrix} \dashrightarrow \begin{bmatrix} x & y \\ y & x \end{bmatrix}^{*} \equiv \begin{bmatrix} x & -y \\ -y & x \end{bmatrix} \in A_{+}(2;\mathbb{C}),$$

and

$$A_{-}(2;\mathbb{C}) \ni \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \dashrightarrow \begin{bmatrix} x & -y \\ y & x \end{bmatrix}^{*} \equiv \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \in A_{-}(2;\mathbb{C}).$$

Remark: This map reduces to conjugation of complex numbers when reduced to real, division, commutative subalgebra $\mathbb{C}$ of anticirculant matrices of $M_2(\mathbb{R})$ i.e. when reduced to

$$\mathbb{C} = \left\{ \begin{bmatrix} x & -y \\ y & x \end{bmatrix},\ x, y \in \mathbb{R} \right\}.$$

In the case of $n > 2$ the $*$-map *is no more linear*. Therefore we call it a *quasi-involution*.

EXAMPLE: Let

$$F = \begin{bmatrix} x_0 & -x_2 & -x_1 \\ x_1 & x_0 & -x_2 \\ x_2 & x_1 & x_0 \end{bmatrix},$$

Naturally

$$M_{00} = M_{11} = M_{22} = x_0^2 + x_1x_2,$$

$$M_{01} = M_{12} = M_{20} = x_2^2 + x_0x_1,$$

$$M_{02} = M_{10} = M_{21} = x_1^2 - x_0x_2,$$

$$F^{*} = \begin{bmatrix} M_{00} & -M_{10} & M_{20} \\ -M_{01} & M_{11} & M_{20} \\ M_{02} & -M_{12} & M_{22} \end{bmatrix}$$

Hence we have $(d \equiv \det F)$

$$\begin{bmatrix} x_o & -x_2 & -x_1 \\ x_1 & x_o & -x_2 \\ x_2 & x_1 & x_o \end{bmatrix} \begin{bmatrix} M_{00} & -M_{10} & M_{20} \\ -M_{01} & M_{11} & M_{20} \\ M_{02} & -M_{12} & M_{22} \end{bmatrix} = \begin{bmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d \end{bmatrix},$$

independently of the value of $\det F$.

OBSERVATION B: Note that the modulus squared of complex numbers is equal to

$$\det \begin{bmatrix} x & -y \\ y & x \end{bmatrix} = \textit{surface of the square spanned by columns of}$$

the anticirculant matrix representing that complex number.

Let then now $F \in A_{\mp} \subset M_n(\mathbb{C})$. A natural generalization of the above *"two-dimensional volume"* of complex number is the n dimensional **volume of a quasinumber** $F \in A_{\mp} \subset M_n(\mathbb{C})$ i.e. determinant of F. Obviously, hyperbolon and elliptic algebras become immediately *"volume-composition"* algebras with quite a lot formulas - resembling complex number system - surviving in an appropriate form; for example

$$A^{-1} \equiv \frac{A^{\bullet}}{A^{\bullet}A} \equiv \frac{A^{\bullet}}{AA^{\bullet}} \; ; \; \forall\, A \in Gl(n;\mathbb{C}).$$

Of special interest become then groups of nonzero-volume quasinumbers.

V. GROUPS OF NONZERO-VOLUME QUASINUMBERS

Consider a circulant matrix

$$H(x) = \begin{bmatrix} h_o(x) & h_{n-1}(x) & \dots\dots\dots h_1(x) \\ h_1(x) & h_o(x) & \dots\dots\dots h_2(x) \\ \dots & \dots & \dots \\ h_{n-1}(x) & h_{n-2}(x) & \dots\dots\dots h_0(x) \end{bmatrix},$$

where

$$h_i : \mathbb{C} \longrightarrow \mathbb{C} \; ; \; i \in Z'_n \; ;$$

$$h_i(x) = \sum_{k=0}^{\infty} \frac{x^{nk+i}}{(nk+i)!} .$$

form the set $\left\{ h_i \right\}_{i \in Z'}$ of *hyperbolic functions* .

Consider also an anti-circulant matrix F

$$F(x) = \begin{bmatrix} f_o(x) & -f_{n-1}(x) & \dots\dots\dots -f_1(x) \\ f_1(x) & f_0(x) & \dots\dots\dots -f_2(x) \\ \dots & \dots & \dots \\ f_{n-1}(x) & f_{n-2}(x) & \dots\dots\dots f_0(x) \end{bmatrix},$$

where $f_i : \mathbb{C} \longrightarrow \mathbb{C} \; ; \; i \in Z'_n \; ;$

$$f_i(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{nk+i}}{(nk+i)!} .$$

form the set $\left\{ f_i \right\}_{i \in Z_n}$ of *elliptic functions* .

Note then that

$$\kappa^{l} f_{l}(x) \equiv h_{l}(\kappa x), \; l \in Z'_{n}$$

and that the following holds:

Observations:

I. $\boxed{\det H(x) = 1 \quad \forall\, x \in \mathbb{C}}$

II. $\boxed{\det F(x) = 1 \quad \forall\, x \in \mathbb{C}}$

Proof: Observations I. & II. follow immediately from the **identities**

$$\boxed{\exp\{x\gamma_{+}\} \equiv H(x) \quad \& \quad \exp\{x\gamma_{-}\} \equiv F(x)}$$

after using the formula

$$\det\left\{\exp A\right\} = \exp\{\mathrm{Tr}A\},$$

and noting that $\mathrm{Tr}\ \gamma_{\pm} = 0.$

Remark: We can prove I. & II. also in the following easy way :

$$\det H(x) = \prod_{m \in Z'_{n}} \left\{ \sum_{l \in Z'_{n}} \omega^{lm} h_{l}(x) \right\} =$$

$$= \prod_{m \in Z'_{n}} \sum_{k \in Z'_{n}} \left\{ 1/n \sum_{l \in Z'_{n}} \omega^{l(m-k)} \right\} \exp(\omega^{k} x) =$$

$$= \prod_{m \in Z'_{n}} \sum_{k \in Z'_{n}} \delta_{mk} \exp(\omega^{k} x) = 1.$$

and for $F(x)$ we have correspondingly

$$\left\{ \lambda_{k} = \sum_{s \in Z'_{n}} f_{s} \left[\exp\left[\frac{i\pi(2k+1)}{n}\right]\right]^{s} \right\}_{k \in Z'_{n}}$$

$$\det F(x) = \prod_{m \in Z'_{n}} \left\{ \sum_{s \in Z'_{n}} \kappa^{(2m+1)s} f_{s}(x) \right\} =$$

$$= \prod_{m \in Z'_{n}} \sum_{s \in Z'_{n}} \left\{ \frac{\kappa^{s}}{n} \sum_{k \in Z'_{n}} \omega^{-ks} \kappa^{(2m+1)s} \right\} \exp(\kappa\omega^{k} x) =$$

$$= \prod_{m \in Z'_{n}} \sum_{k \in Z'_{n}} \left\{ 1/n \sum_{s \in Z'_{n}} \omega^{s(m-k)} \right\} \exp(\kappa\omega^{k} x) =$$

$$= \prod_{m \in Z'_{n}} \sum_{k \in Z'_{n}} \delta_{mk} \exp(\kappa\omega^{k} x) = 1\,.$$

due to the identity

$$1/n \sum_{s \in Z'_{n}} \omega^{s(m-k)} \equiv \delta_{mk}.$$

Consider now two hyperbolic matrices $H(x)$ & $H(y)$, then

$$\text{(H)} \qquad \boxed{H(x)H(y) = H(x+y), \quad \forall\, x,y \in \mathbb{C}}$$

Proof: The thesis follows from the identity:

$$\sum_{l=0}^{n-1} h_{l}(x) h_{k-l}(y) \equiv h_{k}(x+y) \quad ; \quad l,k \in Z'_{n}\,.$$

Similarly for $F(x)$ we have

$$\text{(F)} \qquad \boxed{F(x)F(y) = F(x+y), \quad \forall\, x,y \in \mathbb{C}}$$

Proof: The thesis follows from identities

$$(*) \qquad \sum_{l=0}^{n-1} h_l(x)h_{k-l}(y) \equiv h_k(x+y) \quad ; \quad l,k \in Z'_n ,$$

$$\kappa^l f_l(x) \equiv h_l(\kappa x), \; l \in Z'_n ,$$

from which we get

$$(\#) \qquad \sum_{l=0}^{n-1} f_l(x)f_{k-l}(y) \equiv f_k(x+y) \quad ; \quad l,k \in Z'_n .$$

Observations (H) & (F) follow immediately from identities

$$\exp\{x\gamma_+\} \equiv H(x) \quad \& \quad \exp\{x\gamma_-\} \equiv F(x) ,$$

and thus could be used to derive quickly the identities (*) & (#). Note also that due to

$$h_k(0) = 1/n \sum_{s=0}^{n-1} \omega^{-sk}\exp(\omega^s 0) = \delta_{k0}$$

$$f_s(0) = \kappa^{-1}/n \sum_{k=0}^{n-1} \omega^{-ks}\exp(\omega^k \kappa 0) = \delta_{s0} \quad ; \; k,s \in Z'_n ,$$

we have

$$H(0) = F(0) \equiv \mathbb{I},$$

hence we conclude that

$$GH_n(\mathbb{C}) \equiv \left\{ H(x) \; ; \; x \in \mathbb{C} \right\} \quad \& \quad GF_n(\mathbb{C}) \equiv \left\{ F(x) \; ; \; x \in \mathbb{C} \right\}$$

form the abelian subgroups of $SL(n;\mathbb{C})$ [16, 17]. Naturally

$$H(x)^* = H(x)^{-1} \equiv H(-x) \quad \& \quad F(x)^* = F(x)^{-1} \equiv F(-x) .$$

For $n = 2$ we recover the well known cases

$$GF_2(\mathbb{R}) = \left\{ \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}, \quad x \in \mathbb{R} \right\} \equiv SO(2),$$

$$GH_2(\mathbb{R}) = \left\{ \begin{bmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{bmatrix}, \; x \in \mathbb{R} \right\} \equiv O(1,1.).$$

The group properties of sets of matrices $GH_n(\mathbb{C})$ & $GF_n(\mathbb{C})$, could be used to derive the generalized de Moivre formula [15]

$$(dMh_n) \qquad \left[\sum_{l \in Z'_n} h_l(x) \right]^k = \sum_{l \in Z'_n} h_l(kx),$$

$$(dMf_n) \qquad \left[\sum_{l \in Z'_n} \kappa^{-l} f_l(x) \right]^k = \sum_{l \in Z'_n} \kappa^{-l} f_l(kx),$$

Proof: Because of

$$\kappa^l f_l(x) \equiv h_l(\kappa x), \; l \in Z'_n ,$$

it is sufficient to prove only the generalized de Moivre formula (dMh_n) . Use then

$$\sum_{l=0}^{n-1} h_l(x)h_{k-l}(x) \equiv h_k(2x) \quad ; \quad l,k \in Z'_n ,$$

in order to see that $\sum_{k \in Z_n'} h_k(2x) \equiv$

$$\equiv \sum_{k,l \in Z_n'} h_l(x) h_{k-l}(x) = \sum_{l \in Z_n'} h_l(x) \sum_{k \in Z_n'} h_{k-l}(x) = \sum_{l \in Z_n'} h_l(x) \sum_{k \in Z_n'} h_k(x)$$

which is (dMh_2) formula to start with a recurrence reasoning.

The generalized de Moivre formulas just derived written with help of hyperbolons & elliptons reflect simply the group property of sets $GH_n(\mathbb{C})$ & $GF_n(\mathbb{C})$ of the corresponding quasinumbers; namely de Moivre identities may be written as.

$$H(x)^k \equiv H(kx) \quad \& \quad F(x)^k \equiv F(kx).$$

This would render us with a simple prescription of taking roots of ellipton s and hyperbolons of nonzero volume.

For that to do we shall introduce (see [8]) the notion of the polar form of quasinumber of nonzero volume.

VI. POLAR FORM OF NONZERO-VOLUME QUASINUMBERS

Let us recall that nonzero-volume quasinumbers form the following groups:

Notation:

$$GH_n(\mathbb{C}) := \left\{ h = \sum_{l \in Z_n'} h_l \gamma_+^l \ ; \ h_l \in \mathbb{C} \ ; \ \det h \neq 0 \right\},$$

$$GF_n(\mathbb{C}) := \left\{ f = \sum_{l \in Z_n'} f_l \gamma_-^l \ ; \ f_l \in \mathbb{C} \ ; \ \det f \neq 0 \right\}.$$

Both types of nonzero-volume quasinumbers considered, can be represented in polar form i.e. we have :

OBSERVATION: $\forall \ X \in GH_n(\mathbb{C}) \ \ \exists! \ (\varphi_0, \varphi_1, \ldots, \varphi_{n-1}) \in \mathbb{C}^n$

$$X = \prod_{s \in Z_n'} H_s(\varphi_s),$$

where

$$H_s(\varphi) := \exp\left\{\varphi \gamma_+^s\right\}.$$

and $\forall \ Y \in GF_n(\mathbb{C}) \ \ \exists! \ (\varphi_0, \varphi_1, \ldots, \varphi_{n-1}) \in \mathbb{C}^n$

$$Y = \prod_{k \in Z_n'} F_k(\varphi_k),$$

where

$$F_k(\varphi) := \exp\left\{\varphi \gamma_-^k\right\}.$$

Proof: Denote by $\lambda_k(X)$ the k-th eigenvalue of X

It is then enough to notice that $\lambda_k(\varphi)$; $k \in Z'_n$,

$$\lambda_k(\varphi) \equiv \exp(\omega^k \varphi) = \sum_{k=0}^{n-1} \omega^{kl} h_l(\varphi) ; \quad k \in Z'_n ,$$

is the k-th eigenvalue of $H(\varphi)$ volume-one hyperbolon and in general $\overset{(s)}{\lambda}_k(\varphi)$; $s,k \in Z'_n$,

$$\overset{(s)}{\lambda}_k(\varphi) \equiv \exp(\omega^{ks}\varphi) = \sum_{l=0}^{n-1} \omega^{skl} h_l(\varphi) ; \quad s,k \in Z'_n$$

is the k-th eigenvalue of $H_s(\varphi)$ volume-one hyperbolon.

Both X and $\left\{H_s(\varphi_s)\right\}_{s \in Z'_n}$ hyperbolons i.e. circulant matrices can be diagonalized simultaneously via appropriate inner automorphism as they have the same set of eigenvectors.

Namely, for simultaneous diagonalization we use an inner automorphism given by the matrix $C \equiv \left[\omega^{ks}\right]$; $s,k \in Z'_n$

$$C \equiv \left[\omega^{ks}\right] = \begin{bmatrix} 1 & 1 & 1 \ldots\ldots\ldots\ldots 1 \\ 1 & \omega & \omega^2 \ldots\ldots\ldots\ldots \omega^{n-1} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} \ldots\ldots \omega^{(n-1)^2} \end{bmatrix},$$

k-th column of which is just the k-th eigenvector of γ_+ .

After diagonalization of

$$X = \prod_{s \in Z'_n} H_s(\varphi_s),$$

we get the set of equations

$$\lambda_k(X) = \prod_{s \in Z'_n} \overset{(s)}{\lambda}_k(\varphi_s) ; \quad k \in Z'_n ,$$

from which we get

$$b_k \equiv \mathrm{Ln}\, \lambda_k(X) = \sum_{s \in Z'_n} \omega^{ks}\varphi_s \quad ; \quad s,k \in Z'_n .$$

This set of equations obviously has unique solution

$$(\varphi_0, \varphi_1, \ldots, \varphi_{n-1}) \in \mathbb{C}^n$$

because the matrix $C \equiv \left[\omega^{ks}\right]$; $s,k \in Z'_n$ is invertible as

$$\det\left[\omega^{ks}\right] = i^{-(n-1)(n+2)}\sqrt{n} .$$

$1/\sqrt{n}\, C$ is unitary *for any* n because of

$$1/n \sum_{s \in Z'_n} \omega^{s(m-k)} \equiv \delta_{mk}$$

Analogously we prove the part of the thesis for nonzero-volume anticirculant matrices Y i.e. for nonzero-volume "elliptons''. It is enough to use in the above proof the identity

$$\kappa^l\, f_l(\varphi) \equiv h_l(\kappa\varphi), \quad s \in Z'_n .$$

in order to obtain the corresponding proof for "elliptons".

Namely, we notice (using the above identity) that

$$\varepsilon_k(\varphi) \equiv \exp\left[(\kappa\omega)^k\varphi\right] = \sum_{l \in Z'_n} \kappa^{(2k+1)l} f_l(\varphi) ; \quad k \in Z'_n$$

is the k-th eigenvalue of $F(\varphi)$ volume-one ellipton and in general $\overset{(s)}{\varepsilon}_k(\varphi)$; $s,k \in Z'_n$,

$$\overset{(s)}{\varepsilon}_k(\varphi) \equiv \exp\left[(\kappa\omega)^{ks}\varphi\right] = \sum_{l \in Z'_n} \kappa^{(2k+1)sl} f_s(\varphi) \; ; \quad ; \; s,k \in Z'_n ,$$

is the k-th eigenvalue of $F_s(\varphi)$ volume-one ellipton. Of course Y and $\left\{F_s(\varphi_s)\right\}_{s\in Z'_n}$ ellipTons i.e. anticirculant matrices can be diagonalized simultaneously via appropriate inner automorphism as they have the same set of eigenvectors.

After diagonalization of

$$Y = \prod_{s \in Z'_n} F_s(\varphi_s),$$

we get the set of equations

$$\varepsilon_k(Y) = \prod_{s \in Z'_n} \overset{(s)}{\varepsilon}_k(\varphi_s) \; ; \; k \in Z'_n ,$$

from which we get

$$a_k \equiv \mathrm{Ln}\, \varepsilon_k(Y) = \sum_{s \in Z'_n} (\kappa\omega)^{ks} \varphi_s \; ; \; s,k \in Z'_n .$$

This set of equations obviously has **unique** solution

$$(\varphi_0, \varphi_1, \ldots, \varphi_{n-1}) \in \mathbb{C}^n$$

because the matrix $G \equiv \left[\kappa^{(2k+1)s}\right]$; $s,k \in Z'_n$ is invertible as

$$\det\left[\kappa^{(2k+1)s}\right] = \det\left[\omega^{ks}\kappa^{s}\right] = \det\left[\omega^{ks}\right]\kappa^{n(n-1)/2}.$$

The matrix G was already obtained in [8]. Note also, that $G^{-1} = 1/n\, G^{*T}$ for arbitrary n, therefore $1/\sqrt{n}\, G$ *is unitary for arbitrary* n, because for $\forall\, n \in \mathbb{N}$ we have :

$$1/n \sum_{j \in Z'_n} \kappa^{(2l+1)j} \kappa^{-(2j+1)k} = \delta_{lk} .$$

The polar decomposition allows us to find immediately the

ROOTS: $\forall\, X \in GH_n(\mathbb{C}) \;\; \exists!\, (X_k)_{k \in Z'_N} \subset GH_n(\mathbb{C})$; $X_k^N = X$.

$\forall\, Y \in GF_n(\mathbb{C}) \;\; \exists!\, (Y_k)_{k \in Z'_N} \subset GF_n(\mathbb{C})$; $Y_k^N = Y$.

Proof: Use the polar decomposition and the fact that phase factors $H_s(\varphi) \equiv \exp\left\{\varphi\, \gamma_+^s\right\}$ & $F_k(\varphi) \equiv \exp\left\{\varphi\, \gamma_-^k\right\}$, $s,k \in Z'_n$ of this decomposition are not periodic functions of their complex argument φ. Hence

$$X_k = v_k \prod_{s=1}^{n-1} H_s(\varphi_s/n), \quad k \in Z'_n$$

where v_k is the k-th root of the X quasinumber volume.

The same holds for ellipTons.

VII. FINAL REMARKS

I. *In this remark we indicate the relevance of hyperbolon quasi-numbers to Potts models.*

Let $\left\{ (s_{lk}) \ ; \ s_{lk} \in Z_n \right\}$ denotes the set of states on the p×q torus lattice (p-rows, q-columns), where Z_n is the cyclic group of n-th roots of unity. The Z_n - Potts model is then defined by the expression for the total energy E of the system i.e.

$$\frac{E[(s_{lk})]}{kT} = a \sum_{l,k=1}^{p,q} (s^{-1}_{lk} s_{l,k+1} + s^{-1}_{l,k+1} s_{lk}) + b \sum_{l,k=1}^{p,q} (s^{-1}_{lk} s_{l+1,k} + s^{-1}_{l+1,k} s_{lk}) .$$

The transfer matrix M for this model is then of the form [2] :

$$M = AB,$$

where

$$B = \exp \left[b \sum_{k=1}^{p} (Z^{-1}_k Z_{k+1} + Z^{-1}_{k+1} Z_k) \right],$$

and

$$W(X_k) = \sum_{l \in Z'_n} \lambda_l (a) X^l_k ,$$

$$\lambda_l = \exp(2a \ \mathrm{Re}(\omega^l)),$$

$$\hat{a} = (\lambda_{j-1}),$$

and

$$X_k = I\otimes \ldots \otimes I \otimes \gamma_1 \otimes I \otimes \ldots \otimes I \quad (p - \text{terms}),$$

$$Z_k = I\otimes \ldots \otimes I \otimes \gamma_3 \otimes I \otimes \ldots \otimes I \quad (p - \text{terms}),$$

$$\gamma_1 = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & 0 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix},$$

$$\gamma_3 = \left[\omega^i \delta_{ij} \right]; \ i,j \in Z'_n .$$

Matrices γ_1, γ_3 and $\gamma_2 = \left[\omega^i \delta_{i+1,j} \right]$; $i,j \in Z'_n$ for n = 2 become the well known Pauli matrices. For arbitrary n any two matrices of the three γ_1, γ_2, γ_3 generate the generalized Pauli algebra, the building block for construction of all generalized Clifford algebras [6].

It is easy to notice that the generators γ_1 & γ_3 of generalized Pauli algebra are "dual" in the following sense:

Let

$$C \equiv 1/\sqrt{n} \left[\omega^{ks} \right] \equiv 1/\sqrt{n} \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & \omega & \omega^2 & \ldots & \omega^{n-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \ldots & \omega^{(n-1)^2} \end{bmatrix},$$

Define

$$A^D := C^{-1} AC$$

for any circulant matrix A . Then

$$\gamma_3 = \gamma_1^D .$$

Naturally both γ_1 & γ_3 generate the isomorphic hyperbolon algebras. Namely

$$A_+ := \left\{ h = \sum_{l \in Z_n'} h_l \gamma_1^l \; ; \; h_l \in \mathbb{C} \right\},$$

and

$$A_+^D := \left\{ h^D = \sum_{l \in Z_n'} h_l \gamma_3^l \; ; \; h_l \in \mathbb{C} \right\}.$$

The "dual" representation A_+^D corresponds to the choice the authors of [8] made for their "multicomplex numbers".

This altogether enables us to make the following observation :

Observation:

The transfer matrix M for Z_n- Potts model is the product of two hyperbolon numbers A & B with the latter being represented in the dual representation to that of the former.

The detailed study of consequences of this remarkable fact we leave for the subsequent paper.

II. In the second remark we repeat after [8], (for the sake of completeness), that quasi-number systems considered in this paper have already appeared in various contexts of mathematical physics as incompletely illustrated by [5] [19].

This concerns such vast area of research as "Quantum Groups" (Drinfeld), nuclear physics ('T Hooft), quantum mechanics on a finite phase space (Balian, Itzykson) already investigated in seventies by T.S. Santhanam [20] and naturally statistical physics.

III. Due to the statistical analogy and efficiency of the thermodynamics formalism for fractals and multifractals [21] (vide Potts chains thermodynamics) hyperbolons might be perhaps a good tool also for fractals as they are for code theory [22]. In this connection note the intrinsic relations between code theories and some descriptions of fractal objects .

A more detailed review of the area of application of quasi-number will be given elsewhere.

IV. The remarkable interference of ideas from fractal theory, number theory (stochastic properties of numbers [23]), and thermodynamics [21] might provide us with astonishing perspectives on the grounds of the so called Finite Dimensional Quantum Mechanics started already by Herman Weyl in his famous book [24]. Namely, identification of unitary transformations in Weyl's commutation relations

$$VU = \exp\{2\pi i/n\}UV,$$

with corresponding hyperbolon quasinumbers V & U allows us to interpret the generalized Pauli algebra as the Weyl's quantum mechanics on cyclic group Z_n [24] (see also [20] & [19]). The Heisenberg group then is identical with generalized Dirac or Pauli group - in Clifford algebra language [3].

Hyperbolons V & U satisfying Weyl's commutation relations are dual i.e. one is the other's image under the transformation

$$A \dashrightarrow A^D := C^{-1}AC ,$$

where $C \equiv 1/\sqrt{n}\left[\omega^{ks}\right]$ is now to be interpreted as finite Fourier transform matrix between momentum and position representations of Weyl's Z_n- space quantum mechanics [20].

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NOTES AND DISCUSSIONS

CLIFFORD ALGEBRAS. TOWARDS A COMMON LANGUAGE FOR MATHEMATICIANS AND PHYSICISTS. SOME SUGGESTED NOTATIONS

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Abstract

Based on the discussion on this subject that took place in the II Workshop on Clifford Algebras (Montpellier, 1989), a more detailed account of the proposals is given here. It is intended to be a constructive step towards a unified notation, with the aim of promoting the use of geometric Clifford algebra in all domains of physics, and, at the same time, maintaining a close connection with higher mathematics.

1. Introduction.

Why Clifford Algebra Seems to be a Babel Tower?

At present there are a great variety of competing ideas concerning the role and nature of Clifford algebras. Their existence is due both to the rather old roots of the subject (1878)[1] and to its very special development. Clifford's geometric algebra has not developed steadily from its inception by Grassmann and Clifford and Lipschitz, and does not occupy a "thematic" position in the field of mathematics, not even in the more restricted field of "physical mathematics". It was neglected for a long time, as was the whole of Grassmann's work, and it was only after Dirac's discovery of the relativistic equation for the electron that the subject became a key ingredient of quantum mechanics. Since then, and only disguised as (or confused with) a specific type of matrix algebras, several generations of theoretical physicists have had contact with "Clifford algebra". This has had several negative consequences, as seen from today's perspective.