Newton's 1687 interpolation formula and Stirling numbers

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Abstract

The Newton interpolation formula and divided differences appear helpful and inevitable along with umbra symbolic language in describing properties of general exponential polynomials of Touchard and their possible generalizations. See: the source epos: **Isaak Newton** *Philosophiae Naturalis Principia Mathematica*, Liber III, Lemma V, London (1687).

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1. In the q-extensions realm

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2. Beyond the q-extensions realm

The further consecutive ψ-umbral extension of Carlitz-Gould q-Stirling numbers $\left\{ {n \atop k} \right\}_q$ and $\left\{ {n \atop k} \right\}_q^{\sim}$ is realized two-fold way - one of which leads to a surprise in contrary to the other way.

2.1. The first way

The first "easy way" consists in almost mnemonic sometimes replacement of q subscript by ψ after having realized that in equation (5) we are dealing with the specific case of the so called Comtet numbers [14, 15] (Comtet L. in *Nombres de Stirling generaux et fonctions symtriques* C.R. Acad. Sci. Paris, Series A, 275 (1972):747-750 formula (2) refers to **Wronski**). This array of Stirling-like numbers $\left\{ {n \atop k} \right\}^{\sim}_{\psi}$ - *"alephs de Wronski"* as Comtet refers to it or these Comtet numbers in terminology of Wagner [14, 15] or as a matter of fact these (I. Newton *Philosophiae Naturalis Principia Mathematica*, Liber III, Lemma V, London (1687).)

Newton interpolation coefficients [13] for $e_n, n \geq 0$ i.e. divided differences $[0, 1_\psi, 2_\psi, ..., k_\psi; e_n]$ are defined accordingly as such coefficients - below.

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}^{\sim}_{\psi} \psi_{\underline{k}}(x), \quad n \geq 0, \tag{1}$$

i.e. equivalently (recall that $e_n(x) = x^n, n \geq 0$)

$$\left\{ {n \atop k} \right\}^{\sim}_{\psi} = [0, 1_\psi, 2_\psi, ..., k_\psi; e_n] = \sum_{l=0}^{k} \frac{e_n(l_\psi)}{\psi^{`}_{\underline{k+1}}(l_\psi)}, \quad n \geq 0, \qquad (Newton)$$

where

$$\psi_{\underline{k}}(x) = x(x - 1_\psi)(x - 2_\psi)...(x - [k-1]_\psi)$$

and $\psi^{`}_{\underline{s}}$ denotes the first derivative. Let then $f = \langle f_n \rangle_{n \geq 0}$ be an arbitrary sequence of polynomials. In the following we shall call $S(f; n, k)$ defined below

$$[d_0, d_1, d_2, ..., d_k; f_n] \equiv S(f; \langle d_l \rangle_{l \geq 0}, n, k) \qquad (N - W - C \quad Stirling)$$

the Newton-Wronski-Comptet Stirling numbers (N-W-C for short)- compare with Appendix A.2.

The $\psi^{\sim}$ - Stirling numbers $\left\{ {n \atop k} \right\}^{\sim}_{\psi}$ defined by (10) are specification of $N-W-$

C Stirling array for which we naturally define $\psi^{\sim}$-exponential polynomials $\varphi_n(x,\psi)$ as follows

$$\varphi_n^{\sim}(x,\psi) = \sum_{k=0}^{n}[0,1_\psi,2_\psi,...,k_\psi;e_n]x^k, \quad n \geq 0. \qquad (\psi^{\sim} - exp - pol)$$

Note the trivial but important fact that in the N-W-C Stirling numbers case we are dealing with not equidistant nodes' interpolation in general and note that (*Rescal*) from the subsection 2.2. below is no more valid beyond q-extension case - both with an impact on the way to find out the Dobinski-like formulae - see more below.
As a consequence of (10) we have "for granted" the following extensions of recurrences for Stirling numbers of the second kind:

$$\left\{ {n+1 \atop k} \right\}_\psi^{\sim} = \left\{ {n \atop k-1} \right\}_\psi^{\sim} + k_\psi \left\{ {n \atop k} \right\}_\psi^{\sim}; \quad n \geq 0, k \geq 1, \tag{2}$$

where $\left\{ {n \atop 0} \right\}_\psi^{\sim} = \delta_{n,0}$, $\left\{ {n \atop k} \right\}_\psi^{\sim} = 0$, $k > n$; and the recurrence for ordinary generating function reads

$$G_{k_\psi}^{\sim}(x) = \frac{x}{1-k_\psi} G_{k_\psi - 1}^{\sim}(x), \quad k \geq 1, \tag{3}$$

where naturally

$$G_{k_\psi}^{\sim}(x) = \sum_{n \geq 0} \left\{ {n \atop k} \right\}_\psi^{\sim} x^n, \quad k \geq 1$$

from where one infers that

$$G_{k_\psi}^{\sim}(x) = \frac{x^k}{(1-1_\psi x)(1-2_\psi x)...(1-k_\psi x)} \quad , \quad k \geq 0. \tag{4}$$

Hence we arrive in the standard extended text-book way [22] at the following explicit **new** formula (compare with (2.3) in [15])

$$[0,1_\psi,2_\psi,...,k_\psi;e_n] = \left\{ {n \atop k} \right\}_\psi^{\sim} = \frac{1}{k_\psi!} \sum_{r=1}^{k} (-1)^{k-r} \binom{k_\psi}{r_\psi} r_\psi^n; \quad n \geq k \geq 0, \tag{5}$$

where

$$\sum_{r=1}^{k}(-1)^{k-r}\binom{k_\psi}{r_\psi}r_\psi^n; \quad n,k\geq 0$$

is readily recognized as the ψ-extension of the formula for surjections in its - after inclusion-exclusion principle had been applied - form.

Expanding the right hand side of (13) results in another explicit formula for these ψ-case Newton-Wronski-Comtet array of Stirling numbers of the second kind i.e. we have

$$\left\{ n \atop k \right\}_\psi^\sim = \sum_{1\leq i_1\leq i_2\leq \ldots\leq i_{n-k}\leq k}(i_1)_\psi(i_2)_\psi\ldots(i_{n-k})_\psi; \qquad n\geq k\geq 0 \tag{6}$$

or equivalently (compare with [13, 14])

$$\left\{ n \atop k \right\}_\psi^\sim = \sum_{d_1+d_2+\ldots+d_k=n-k, \quad d_i\geq 0} 1_\psi^{d_1}2_\psi^{d_2}\ldots k_\psi^{d_k}; \quad n\geq k\geq 0. \tag{7}$$

N-W-C case $\psi^\sim$ - **Stirling numbers** of the second kind being defined equivalently by (10), (*Newton*), (14), (15) or (16) yield N-W-C case $\psi^\sim$ - **Bell numbers**

$$B_n^\sim(\psi) = \sum_{k=0}^{n}\left\{ n \atop k \right\}_\psi^\sim = \sum_{k=0}^{n}[0, 1_\psi, 2_\psi, \ldots, k_\psi; e_n], \qquad n\geq 0 \qquad\qquad (B^\sim).$$

Naturally $\exists!$ functional $L^\sim$ such that on the basis of persistent root polynomials $\psi_{\underline{k}}(x)$ it takes the value 1:

$$L^\sim(\psi_{\underline{k}}(x)) = 1, \quad k\geq 0.$$

Then from (10) we get an analog of (3)

$$B_n^\sim(\psi) = L^\sim(x^n) \qquad\qquad (L^\sim).$$

Problem: which distribution the functional $L^\sim$ is related to is an open technical question by now. More - the recurrence for $B_n^\sim(\psi)$ is already quite involved and complicated for the q-extension case (see: the first section)- and no acceptable readable form of recurrence for the ψ-extension case is known to us by now.

Nevertheless after adapting the standard text-book method [23] we have the following formulae for two variable ordinary generating function for $\left\{ {n \atop k} \right\}^{\sim}_{\psi}$ Stirling numbers of the second kind and the ψ-exponential generating function for $B_n^{\sim}(\psi)$ Bell numbers

$$C_{\psi}^{\sim}(x,y) = \sum_{n\geq 0} \varphi_n^{\sim}(\psi, y) x^n, \tag{8}$$

where the ψ- exponential polynomials $\varphi_n^{\sim}(\psi, y)$

$$\varphi_n^{\sim}(\psi, y) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}^{\sim}_{\psi} y^k$$

do satisfy the recurrence (compare with formulas (28) in Touchard's [24] from 1956)

$$\varphi_n^{\sim}(\psi, y) = [y(1 + \partial_{\psi}]\varphi_{n-1}^{\sim}(\psi, y) \qquad n \geq 1,$$

hence

$$\varphi_n^{\sim}(\psi, y) = [y(1 + \partial_{\psi}]^n 1, \qquad n \geq 0.$$

The linear operator ∂_{ψ} acting on the algebra of formal power series is being called (see: [1, 2] and references therein) the "ψ-derivative" as $\partial_{\psi} y^n = n_{\psi} y^{n-1}$.
The $\psi^{\sim}$ - exponential generating function

$$B_{\psi}^{\sim}(x) = \sum_{n\geq 0} B_n^{\sim}(\psi) \frac{x^n}{n_{\psi}!} \qquad (\psi^{\sim} - e.g.f.)$$

for $B_n^{\sim}(\psi)$ Bell numbers - after cautious adaptation of the method from the Wilf's generatingfunctionology book [23] can be seen to be given by the following **new** formula

$$B_{\psi}^{\sim}(x) = \sum_{r\geq 0} \epsilon(\psi, r) \frac{e_{\psi}[r_{\psi} x]}{r_{\psi}!} \tag{9}$$

where (see: [1,2] and references therein)

$$e_{\psi}(x) = \sum_{n\geq 0} \frac{x^n}{n_{\psi}!}$$

while

$$\epsilon(\psi, r) = \sum_{k=r}^{\infty} \frac{(-1)^{k-r}}{(k_\psi - r_\psi)!} \tag{10}$$

and the **new** Dobinski - like formula for the ψ-extensions here now reads

$$B_n^{\sim}(\psi) = \sum_{r \geq 0} \epsilon(\psi, r) \frac{r_\psi^n}{r_\psi!}. \tag{11}$$

The $\psi^{\sim}$-exponential polynomials are therefore given correspondingly by

$$\varphi_n^{\sim}(\psi, x) = \sum_{r \geq 0} \epsilon(\psi, r) \frac{r_\psi^n}{r_\psi!} x^r. \qquad (\psi^{\sim} - exp - pol - II)$$

In the case of Gauss q-extended choice of $\langle \frac{1}{n_q!} \rangle_{n \geq 0}$ admissible sequence of extended umbral operator calculus equations (19) and (20) take the form

$$\epsilon(q, r) = \sum_{k=r}^{\infty} \frac{(-1)^{k-r}}{(k-r)_q!} q^{-\binom{r}{2}} \tag{12}$$

and the **new** N-W-C case $q^{\sim}$-Dobinski formula is given by

$$B_n^{\sim}(q) = \sum_{r \geq 0} \epsilon(q, r) \frac{r_q^n}{r_q!}, \tag{13}$$

which for $q = 1$ becomes the Dobinski formula from 1887 [4]. Note the appearance of re-scaling factor $q^{-\binom{r}{2}}$ in (21). In its absence we would get **not** $q^{\sim}$-Dobinski but q-Dobinski formula

$$B_n(q) = \frac{1}{\exp_q(1)} \sum_{0 \leq k} \frac{k_q^n}{k_q!} \qquad (q - Dobinski)$$

- see [15] and formula (5.28) there coinciding with N-W-C case of Dobinski formula after re-scaling in correspondence with (Rescal) below in subsection 2.2. Correspondingly we would the have **not** $(q^{\sim} - exp - pol)$ formula but $(q - exp - pol)$ formula:

$$\varphi_n(x,q) = \sum_{k=0}^{n} q^{\binom{k}{2}}[0, 1_q, 2_q, ..., k_q; e_n]x^k = \sum_{k=0}^{n} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}_q x^k. \qquad (q-exp-pol)$$

The interpretation problem. ....

In the inversion-dual way to our equation (10) above we define the $\psi^{\sim}$-Stirling numbers of the first kind as coefficients in the following expansion

$$\psi_{\underline{k}}(x) = \sum_{r=0}^{k} \left[\begin{matrix} k \\ r \end{matrix} \right]_{\psi}^{\sim} x^r \tag{14}$$

where - recall $\psi_{\underline{k}}(x) = x(x-1_\psi)(x-2_\psi)...(x-[k-1]_\psi)$. (Attention: see equations (10)-(16) in [8] and note the difference with the present definition). Therefore from the above we infer that

$$\sum_{r=0}^{k} \left[\begin{matrix} k \\ r \end{matrix} \right]_{\psi}^{\sim} \left\{ \begin{matrix} r \\ l \end{matrix} \right\}_{\psi}^{\sim} = \delta_{k,l}. \tag{15}$$

Another natural counterpart to $\psi^{\sim}$-Stirling numbers of the second are ψ^c-Stirling numbers of the first kind defined here down as coefficients in the following expansion (”c” because of cycles in non-extended case)

$$\psi_{\overline{k}}(x) = \sum_{r=0}^{k} \left[\begin{matrix} k \\ r \end{matrix} \right]_{\psi}^{c} x^r \tag{16}$$

where - now $\psi_{\overline{k}}(x) = x(x+1_\psi)(x+2_\psi)...(x+[k-1]_\psi)$. These are to be studied elsewhere.

On interpretation. For possible **unified** combinatorial interpretations of binomial coefficients of both kinds, the Stirling numbers of both kinds and the Gaussian coefficients of the first and second kind - i.e for the specific choices of $\psi = \langle \frac{1}{n_\psi !} \rangle_{n \geq 0}$ - see [27, 28]. As for q-analogue of Stirling cycle numbers see [29] and Sect. 5.3. in [30]. The problem of eventual combinatorial interpretation of other ψ-extensions (vide Fibonomial - for example) - remains opened.

2.2. The second way.

.... The selective comparison of the presented umbral extensions of Stirling numbers, Bell numbers and Dobinski-like formulas with other existing extensions (as well as relevant information in brief) serves the purpose of seeking analogies and is to be find in the Appendix that follows now.

Appendix - for remarks, discussion and brief comparative review of ideas.

A.1. Notation.

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A.2. Discussion, remarks, questions.

.... Then due to the recursion for Stirling numbers of the second kind and the identity (operators act on P)

$$\hat{x}(D+1) \equiv \frac{1}{\exp(x)}(\hat{x}D)\exp(x)$$

one defines in GHW - algebra manner the **exponential polynomials**

$$n \geq 0, \qquad \varphi_n(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k \qquad (ExPol)$$

introduced by Acturialist J.F. Steffensen [39, 40] (see: Bell‘s "Exponential polynomials" in umbra-symbolic language [41] p. 265 and his symbolic formula (4.7) for now Bell numbers). These exponential polynomials were substantially investigated by Touchard in Blissard umbra-symbolic language [24]. Here now comes the GHW-definition [38] of these basic polynomials

$$\varphi_n(x) = \frac{1}{\exp(x)}(\hat{x}D)^n \exp(x)$$

resulting in the formula which becomes Dobinski one for $x = 1$ i.e.

$$\varphi_n(x) = \frac{1}{\exp(x)} \sum_{0 \leq k} \frac{k^n x^k}{k!}.$$

Note: The q-case as well as ψ-case **formal** mnemonic counterpart formulae are automatically arrived at with the mnemonic attaching of q or ψ indices to nonnegative numbers [1, 2] - vide:

$$\varphi_n(x,\psi) = \frac{1}{\exp_\psi(x)} \sum_{0\leq k} \frac{k_\psi^n x^k}{k_\psi!} \tag{17}$$

which for $\psi = \langle \frac{1}{n_q!} \rangle_{n\geq 0}$ and $x = 1$ becomes the well known q-Dobinski formula as of course $\varphi_n(x = 1, q) = B_n(q)$ - see in [15] the formula (5.28) and note that this is **not** $q^{\sim}$-Dobinski formula (22) as noticed right after (22). As for eventual second way's ψ-extensions beyond the q-extension case where the *rescaling* does not take place - we are left with an opened problem how to eventually find the way to get round this inspiring obstacle. Perhaps instead of the second beyond the q-extension way we might follow Alexander the great in his Gordian Knot problem solution and define $S(\psi, n, k)$ as follows (whenever one may prove that the object being defined is really a polynomial):

$$\varphi_n(x,\psi) = \sum_{k=0}^{n} S(\psi,n,k)x^k = \frac{1}{\exp_\psi(x)} \sum_{0\leq k} \frac{k_\psi^n x^k}{k_\psi!}. \qquad (S(\psi) - exp - pol)$$

An alternative good idea perhaps would be an attempt to ψ-extend the celebrated Newton interpolation formula (use ∂_ψ instead D, then exp_ψ instead of exp and then you will be faced with ψ-Leibniz rule application problem though... see [1, 2, 33] for Leibnitz rules). Let us then make - also for the sake of comparison with existing knowledge - let us then make us wonder on the intrinsic presence and assistance of Newton interpolation which corresponds to the first "easy" way as described in Subsection 2.1.
The intrinsic presence and assistance of Newton interpolation formula in derivation of Dobinski formula for exponential polynomials and their binomial analogues was underlined and used in [42] for specific presentation of the $q = 1$ case from the umbral point of view of the classical finite operator calculus. In [42] a Dobinski-like formula was derived being as a matter of fact the particular ("binomial") case of formula (30) from Touchard's 1956 year paper [24]. In more detail. Choosing any binomial polynomial sequence $\langle b_n \rangle_{n\geq 0}$ consider its Newton interpolation formula

$$b_n(x) = \sum_{k=0}^{n} [0, 1, 2, ..., k; b_n] x^{\underline{k}}.$$

Then apply an umbral operator sending the binomial basis $\langle x^{\underline{n}} \rangle_{n\geq 0}$ of delta operator Δ to the binomial basis $\langle x^n \rangle_{n\geq 0}$ of delta operator D. Then use

$$[0,1,2,...,k;b_n] = \frac{\Delta^k b_n|_{x=0}}{k!} = \sum_{l=0}^{k} \frac{(-1)^{k-l} b_n(l)}{(k-l)!l!} \quad (Newton - Stirling)$$

so as to arrive (thanks to binomial convolution) at Dobinski like formula from [42] i.e.

$$b_n(\varphi(x)) = \frac{1}{\exp(x)} \sum_{k=0}^{\infty} \frac{b_n(k)x^k}{k!},$$

where φ is the umbral symbol satisfying [24]

$$\varphi_{n+1} = x(\varphi+1)^n, \quad \varphi^{\underline{k}} = x^k. \qquad (Touchard)$$

In order to see that this is just the particular ("binomial") case of umbra-symbolic formula (30) from Touchard's 1956 year paper [24] just choose in Touchard formula (30) the *arbitrary* polynomial f to be any binomial one $b_n = f$. Then $f(\varphi) = b_n(\varphi) = b_n(\varphi(x))$ is binomial also and we have

$$f(\varphi) = \frac{1}{\exp(x)} \sum_{k=0}^{\infty} \frac{b_n(k)x^k}{k!}. \qquad (Dobinski - Touchard)$$

Equidistant nodes Newton's interpolation array of coefficients $[0,1,2,...,k;b_n]$ - here the connection constants of the general exponential polynomial $p_n(x) = b_n(\varphi(x))$ are to be called in the following the **Newton-Stirling** numbers of the second kind and are consequently given by

$$p_n(x) = \frac{1}{\exp(x)} \sum_{k=0}^{\infty} \frac{b_n(k)x^k}{k!} = \sum_{k=0}^{n} [0,1,2,...,k;b_n]x^k, \qquad (N-S-Dob)$$

where $\langle b_n \rangle_{n\geq 0}$ is any sequence of polynomials. These are - in their turn - the special case of N-W-C Stirling numbers.

Coherent States' Example I. Take the $b_m(x) = f(x)$ in the (*Dobinski-Touchard*) formula to be of the form resulting from normal ordering problem (see A.3.II. - below) i.e. let (see: [10])

$$f(x) = b_{ns}(x;r,s) = \prod_{j=1}^{n} [x + (j-1)(r-s)]^{\underline{s}}$$

Then we get (2.8) from [10] i.e.

$$[0,1,2,...,k;b_{ns}(...;r,s)] = \frac{1}{k!} \sum_{l=s}^{k} (-1)^{k-l} b_n s(l;r,s) \binom{k}{l} \equiv S_{r,s}(n,k)$$

becomes the definition of the generalized Stirling numbers (see A.3.II. - below), which appear to be special case of general Newton-Stirling numbers of the second kind. (Here $b_{ns}(...;r,s)(x) = b_{ns}(x;r,s)$.) Naturally the Dobinski-like formula (2.1) from [10] for exponential polynomials determined by $[0,1,2,...,k;b_{ns}(.;r,s] = S_{r,s}(n,k)$ is special case of (N-S-Dob) Dobinski-like formula with counting adapted to the choice $f = b_{ns}$. Along with Bell numbers' sequence or Bessel numbers's sequence this special case of Newton-Bell numbers' sequence

$$B_{r,s}(n) = \sum_{l=s}^{ns} S_{r,s}(n,k)$$

is a moment sequence [43].

Example II The next example of Newton-Stirling numbers $d_{n,k}$ comes from the paper [44] on interpolation series related to the Abel-Goncharov problem. There the divided difference functional Δ_k is applied to e_n yielding $d_{n,k}$ accordingly:

$$\Delta_k e_n = [0, \frac{1}{k}, \frac{2}{k}, ..., \frac{k-1}{k}, 1; e_n] = d_{n,k}.$$

The general rules for Newton-Stirling arrays allow us to notice that

$$d_{n,k} = [0, \frac{1}{k}, \frac{2}{k}, ..., \frac{k-1}{k}, 1; e_n] = \frac{k^k}{k!} \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} \frac{r^n}{k^n}; \quad n \geq k \geq 0,$$

hence for corresponding exponential polynomials we have

$$\varphi_n(x) = \sum_{k=0}^{n} \frac{k^k}{k!} \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} \frac{r^n}{k^n} x^k,$$

in accordance with the fact [44] that $k^{n-k} d_{n,k} = \left\{ {n \atop k} \right\}$. Derivation of the Dobinski-like formula we leave as an exercise.

On ψ-extension. A ψ-extension of the above Touchard's symbolic definition of exponential polynomials would start with the defining formula

$$\varphi_{n+1} = x(\varphi +_\psi 1)^n, \quad \varphi^{\underline{k}} = x^k. \qquad (\psi - Exp - Pol)$$

resulting in analogous umbra-symbolic identities and with corresponding Dobinski-like formula as (35) below, where $b_n = e_n$. Compare these with (10) from where we have for this case of $b_n(x) = e_n(x) = x^n, n \geq 0$ the Newton interpolation formula

$$x^n = \sum_{k=0}^{n} [0, 1_\psi, 2_\psi, ..., k_\psi; e_n] \psi_{\underline{k}}(x), \quad n \geq 0.$$

For the meaning of the ψ-shift "$+_\psi$" see [1, 2, 26, 31, 33]. This we shall develop elsewhere. ....
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A.5.3.Remark The relevance of Schlömilch's work [94] from **1852** to N-W-C Stirling numbers is taken down here with pleasure. Another interesting paper refereeing directly to the original Dobinski's work [4] and Dobinski's point of view is the Fekete's paper [95] from 1999.

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