Fault-tolerant pancyclicity in alternating group graphs

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Abstract

In [Z.-J. Xue, S.-Y Liu, An optimal result on fault-tolerant cycle-embedding in alternating group graphs, Inform. Process. Lett. 109 (2009) 1197–1201] the authors claim that every alternating group graph AG_n is $(2n-6)$ fault tolerant pancyclic. Which means that if the number of faults $|F| \leq 2n-6$, then $AG_n - F$ contains cycles of every length from 3 to $n!/2 - |F|$. Their proof is not complete. They left a few important things unexplained. In this paper we fulfill these gaps and present another proof that AG_n is $(2n-6)$-fault-tolerant pancyclic.

Key words: hamiltonian cycle, pancyclicity, alternating group graph, fault tolerance

1 Introduction

An alternating group graph AG_n, $n \geq 3$ has vertices labeled by even permutations of the set $\{1, ..., n\}$. Two vertices p and q are neighbors if one of them is obtained from the other by rotating three symbols: the first, second, and i-th, for some $i \geq 3$. There are $n!/2$ vertices in AG_n. The graph AG_3 has three vertices 123, 231, and 312, every two are connected. AG_4 is presented in Fig 1. By F we shall denote the set of faulty vertices. In [2] the authors claim that AG_n is $(2n-6)$ fault tolerant pancyclic. Which means that if the number of faults $|F| \leq 2n-6$, then $AG_n - F$ contains cycles of every length from 3 to $n!/2 - |F|$. Their proof, by induction, is not complete. They left a few important things un-

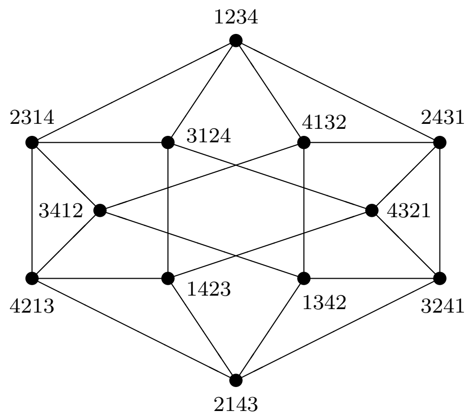

Fig. 1. AG_4

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explained. When proving induction step they decompose AG_n into subgraphs A^1,...,A^n. By induction hypothesis, shortest cycles are in these subgraphs. To obtain longer cycles they take a cycle C already build and extend it into the next subgraphs using so called 4-cycle structures. If the cycle C is contained in one subgraph A^i then it is easy to see that C can be extended into a new subgraph A^j. But the authors do not explain how to find such expanding structure if C goes through more than one subgraph and there are only few subgraphs unvisited. Moreover they do not explain why there are cycles of every length $\ell \geq 3$ going through 4-cycle structure in A^j.

In this paper we fulfill the gaps and present another proof that AG_n, $n \geq 4$, is $(2n-6)$-fault-tolerant pancyclic. We shall prove the theorem by induction. For $n = 4$ the theorem was proven in [1].

Lemma 1 *(see [1]) AG_4 is 2-fault tolerant pancyclic.*

For $n \geq 5$, we shall decompose AG_n into subgraphs and we show that shortest cycles are constructed in subgraphs with smallest number of faults. In order to build longer cycles we first find a cycle in the subgraphs with many faults and extend it to the rest of the graph. In [2] the authors noted that the bound $2n-6$ is optimal, because with $2n-5$ faults we can remove all but one neighbors of a vertex v, so no hamiltonian cycle is possible.

2 Alternating group graph

An alternating group graph AG_n, $n \geq 3$ has vertices labeled by even permutations of the set $\{1, ..., n\}$. The permutation $p = (p_1, ..., p_n)$ is even if it contains the even number of inversions. The inversion is a pair of numbers i, j, $1 \leq i < j \leq n$ such that $p_i > p_j$. For every i, $3 \leq i \leq n$, let g_i^+ be the permutation which rotates symbols in positions 1,2, and i from left to right; and g_i^- be the permutation which rotates these symbols from right to left. Two vertices p and q are connected by an edge if and only if $q = pg_i^+$ or $q = pg_i^-$ for some $i \geq 3$. Observe that if $q = pg_i^+$ then $p = qg_i^-$. There are $n!/2$ vertices in AG_n, and each vertex is connected with $2n-4$ neighbors. The graph AG_n can be divided into subgraphs A^1,...,A^n, each A^i contains vertices with i on the last symbol. The subgraph A^i is isomorphic with AG_{n-1}. We can also divide AG_n according to other position, say k, for some $3 \leq k \leq n-1$. Then A^i contains vertices with i on the k-th position. Note that every two vertices u and v must differ in some symbol $k \geq 3$ and we can decompose AG_n in such a way that u and v are in different subgraphs, and we can always assume that faulty vertices are not in one subgraph. On the other hand we can also divide AG_n, $n \geq 4$, in such a way that two ends of an edge are in one subgraph. This is because they differ only in one position $i \geq 3$. Every vertex $u \in A^i$ is connected with exactly two vertices u' and u'' which are in two different subgraphs outside A^i. We will call the edges (u, u') and

(u, u'') external edges. Other edges we shall call internal. For each internal edge $(u, v) \in A^i$ with $u = (kj...i)$ and $v = (jk'...i)$ there exist adjacent vertices $s = (ik...j)$ and $t = (k'i...j)$ both in A^j which form the 4-cycle (u, s, t, v). We shall say that the edge (u, v) is of color j or that it is connected (by a 4-cycle) with the edge (s, t) in A^j. If a subgraph A^i is of dimension 4 and is isomorphic to AG_4 (see Fig. 1) then there are 4 colors, the edges of each color form a cycle of length 6. For example, the cycle 1234, 4132, 1342, 2143, 1423, 3124 contains edges of color 1. If a subgraph A^i is of dimension 5 then it can be divided into 5 subgraphs A^i_1, A^i_2, A^i_3, A^i_4, A^i_5 (according to the 5-th position). Each of A^i_j is isomorphic to AG_4, and contains 4 colors (all colors except i and j), and edges in each color form a cycle of length 6. Similarly for higher dimensions. AG_n can be divided into n subgraphs A^1,...,A^n according to the last position. Each A^i can be divided into $(n-1)$ subgraphs A^i_1,...,A^i_{n-1} according to the last by one position and so on. But the color of the edge depends only on the first two symbols and is the same in each subgraph. Moreover if an edge (u, v) is in the subgraph A^i_k and is connected with the edge (u', v') in A^j, then the edge (u', v') is in the subgraph A^j_k. There are $(n-2)!$ external edges joining two different subgraphs A^i and A^j. If x and y are two vertices in a subgraph, say A^1, then it is easy to see that we can choose external edges (x, x') and (y, y') in such a way that x' and y' are in two different subgraphs. Moreover if x and y are neighbors and A^j is some other subgraph $j \geq 2$ then we can choose x' and y' to be not in A^j. This is because if x is connected by external edges with x' and x'' in two different subgraphs and y is connected with y' and y'' also in two different subgraphs, and x' and y' are in one subgraph then (x, y, y', x') form a 4cycle structure and there is only one 4-cycle structure for the edge (x, y). By F we shall denote the set of faulty vertices; $f_i = |A^i \cap F|$ denotes the number of faulty vertices in A^i, and $h_i = |A^i - F|$ denotes the number of healthy vertices in A^i.

Lemma 2 *(see [2]) Let A^1,..., A^k be arbitrary string of subgraphs from the decomposition of AG_n, $2 \leq k \leq n$. Each $A^i - F$ is hamiltonian connected and each A^i is connected with A^{i+1} by at least 3 healthy edges. Then for every $s \in A^1$ and $t \in A^k$ there is hamiltonian path connecting s and t in $A^1 \cup ... \cup A^k$.*

We say that a graph is hamiltonian connected if for every two vertices u and v, there is hamiltonian path from u to v.

Lemma 3 *AG_n, $n \geq 3$, is hamiltonian connected if $|F| \leq n - 3$.*

Proof. By induction on n. It is easy to check that the lemma is valid for $n = 3$ or $n = 4$. For $n \geq 5$ let us divide AG_n into subgraphs A^1,...,A^n. We can assume that faulty nodes are not in one subgraph and $f_i \leq n-4 = (n-1)-3$ for all i. Suppose first that u and v are in the same subgraph, say in A^1. By induction hypothesis, there is hamiltonian path P_1 in A^1 connecting u and v. The length of P_1 is $|P_1| = (n-1)!/2 -$

f_1-1, so we can choose $\lfloor((n-1)!/2-f_1-1)/2\rfloor$ separate edges in P_1 and there are only $n-3-f_1$ faulty vertices outside A^1. Hence we can find an edge (x,y) in P_1 with healthy external edges (x,x'), (y,y') going to two different subgraphs. By induction hypothesis, every A^i is hamiltonian connected and every two A^i and A^j are connected by at least $(n-2)!-(n-3)\geq 3$ edges. Hence, by Lemma 2, x' and y' can be connected by a path P_2 going through all vertices outside A^1. The case when u and v are in two different subgraphs follows immediately from Lemma 2. $\square$

3 Extending cycles

Lemma 4 *Consider decomposition of AG_{n+1}, into subgraphs A^1,...,A^{n+1} of dimension n and let A^i and A^k be two of them. and let u, v be arbitrary two vertices in A^i (u and v may form an edge).*

(1) If $n=4$ and $f_i=1$ then there is a hamiltonian path in A^i going from u to v through an edge of color k.

(2) If $n\geq 5$ and f_i, $f_k\leq 1$. Then there is a hamiltonian path P in A^i from u to v which goes through an edge e which is connected with an edge $e'\in A^k$ standing in an faultless subgraph A^k_m of A^k. There is also a path P' of length $|P'|=|P|-1$ with the same property.

(3) If $n\geq 4$ and $f_i=2$ then there is a hamiltonian cycle in A^i going through an edge of color k.

Proof: (1) can be easily proven by examining all cases. To prove (2) consider decomposition of A^i into subgraphs A^i_1,...,A^i_n. We can assume that u and v are in different subgraphs. Similarly as in Lemma 3 we can show that u and v can be connected by a path going through all subgraphs by a hamiltonian path. At least 4 subgraphs contain color k. Thus we can choose a subgraph A^i_m with color k, such that the subgraph A^i_m and A^k_m are free of faults. By (1) or induction hypothesis, we can assume that the path goes by color k in A^i_m. In order to obtain the path P', we omit one vertex in some faultless subgraph other than A^i_m. This is possible because AG_n without faults is panconnected [1].

(3) Proof by induction on n. It is easy to check that the lemma is valid for $n=4$. For $n\geq 5$ let us divide A^i into subgraphs A^i_1,...,A^i_n. We may assume that A^i_1 contains at least one fault. By induction hypothesis, there is hamiltonian cycle C in A^i_1. There is at most one fault outside A^i_1, so we can find an edge (x,y) in C with healthy external edges going to two different subgraphs. Similarly as in the proof of (2) we can extend C by a path P going through the rest of A^i and visiting color k. $\square$

Lemma 5 *For any two edges $e,f\in AG_n$, $n\geq 3$, there exists a hamiltonian cycle going through e and f.*

Proof. By induction on n. It is easy to see that the lemma is true for $n=3$ or $n=4$. Let $n\geq 5$. Decompose AG_n into subgraphs A^1,...,A^n. We may assume that both ends of e (and f) are in one subgraph (e and f may be in different subgraphs). This is because there is a position $j \geq 3$ such that neither e nor f differ in position j.

Case 1. Edges e and f are in one subgraph, say in A^1. By induction hypothesis, there is a hamiltonian cycle C going through e and f. There is a third edge in C which connects C with another subgraph and further with the rest of AG_n.

Case 2. Edges e and f are in different subgraphs, say in e in A^1 and f in A^n. For every i, $1 \leq i \leq n$; we choose two edges e_i and $f_i \in A^i$ such that f_i and e_{i+1} form a 4-cycle; and $e = e_1$, $f_n = f$. By induction hypothesis, there is a hamiltonian cycle in A_i going through e_i and f_i. All these cycles can be connected in one hamiltonian cycle. □

We say that a graph G is k-edge-pancyclic if for every edge e there is a cycle going through e of every length from k to $|G|$.

Lemma 6 *(1) AG_n with $|F| = 0$ and $n \geq 3$ is 3-edge-pancyclic.*

(2) AG_4 with $|F| = 1$ is 5-edge-pancyclic.

(3) AG_n with $|F| = 1$ and $n \geq 5$, is 4-edge-pancyclic. Moreover if an edge e is in faultless subgraph, then there is also a cycle of length 3 going through e.

Proof (1) follows from the symmetry of AG_n. (2) can be easily proven looking through all cases. To prove (3) let us decompose AG_n into subgraphs A^1,...,A^n. We can assume that both ends of e are in one subgraph, say A^1.

Case 1. The faulty vertex is also in A^1. The edge e is connected by 4-cycle with an edge outside A^1. By induction hypothesis, or by (2), there is a cycle going through e of every length from 5 to $(n-1)!/2-1$. Hence there is a cycle C of length $(n-1)!/2-1$ and a cycle C' of length $(n-1)!/2-2$ going through e. We choose an edge f in C (and f' in C'). The edge f is connected with an edge e_2 in another subgraph, say A^2 (f' is connected with an edge e_2' in A^2). From C and e_2 one can build a cycle of length $|C|+2$. Similarly using C' one can build a cycle of length $|C|+1$. There is a cycle in A^2 going through e_2 of every length from 3 to $(n-1)!/2$. Joining this cycle with C we obtain cycles of length from $|C|+3$ to $|C|+(n-1)!/2$. We proceede in a similar maner in order to extend these cycles into the next subgraphs. First we get in A^2 an edge f_2 which has a connection with A^3. By Lemma 5, there is a hamiltonian cycle C_1 in A^2 going through e_2 and an edge f_2 (and C_2' going through e_2' and f_2). Joining C and C_1 we obtain the cycle of length $|C|+(n-1)!/2$ going by f_2. Joining C' with C_1 we obtain the cycle of length $|C|+(n-1)!/2-1$ going by an f_2. Similarly as before we can extend this cycles to the cycles of every length from $|C|+(n-1)!/2+1$ to $|C|+(n-1)!$ and further to the next subgraphs.

Case 2. The faulty vertex is in another subgraph, say in A^n. By (1), there is a cycle going through e of every length from 3 to $(n-1)!/2$. Similarly as in Case 1, we can extend these cycles into subgraphs A^2,...,A^{n-1}. To obtain the longest cycles, we first get a hamiltonian cycle C in A^1 which goes by e end has connection with A^n, this is possible by Lemma 5. In A^n we add the hamiltonian cycle. Next we choose the edge in C with connection with another subgraph and through this connection we extend cycles into the rest of subgraphs. $\square$

Lemma 7 *(1) Let A_1,...,A_k be a sequence of subgraphs of dimension n, $n \geq 4$ without faults; e be an edge in A^1. Then there is a cycle in $A_1 \cup ... \cup A_k$ going through e of every length ℓ, $3 \leq \ell \leq k \cdot n!/2$.*

(2) Additionally let g be an arbitrary edge in A^k. Then there is hamiltonian cycle in $A_1 \cup ... \cup A_k$ going through e and g.

(3) Let A_1,...,A_k be a sequence of subgraphs of dimension n, $n \geq 5$ with at most one fault each; e be an edge in A^1. Then there is a cycle in $A_1 \cup ... \cup A_k$ going through e of every length $\ell \geq 3$.

Proof: (1) Since AG_n is pancyclic there is cycle in A^1 of length ℓ for $3 \leq \ell \leq n!/2$. By symmetry, we can assume that every of this paths goes through e. By Lemma 4, there is a hamiltonian cycle C_1 in A^1 going through e and an edge f connected with A^2 and there is a cycle C_1' of length $|C_1| - 1$ going through e and an edge f' connected with A^2. Similarly as in the proof of Lemma 6 we can show that these cycles can be extended into every length greater than $|C|$.

(2) When we construct the hamiltonian cycle going through all subgraphs, then by Lemma 5, we can choose the cycle in the last subgraph A^k in such a way that it goes through the edge g.

(3) The proof is similar to that of (1). By Lemma 4(2), in A^1 there is a hamiltonian cycle C and a cycle C' shorter by one which goes by e and can be extended into A^2. We use Lemma 4(2) in order to extend cycles into next subgraphs. $\square$

4 Main result

Lemma 8 *AG_5, is 4-fault-tolerant pancyclic. That is, if the number of faults $|F| \leq 4$, then it contains a cycle of every length ℓ from 3 to $60 - |F|$.*

We decompose AG_5 into subgraphs A^1,...,A^5. We can assume that the sequence f_1,...,f_5 is nondecreasing. Since $|F| = 4$ we have $f_1 = 0$ and $f_3 \leq 1$. Shortest cycles of lengths from 3 to $h_1 + h_2 + h_3 \geq 34$. we build in $A^1 \cup A^2 \cup A^3$. First in A_1, then in $A_1 \cup A_2$ and at the end we build a hamiltonian cycle in A_1 and extend it into A_2 and A_3. Next we build the longest cycles.

Case 1. $f_5 = 3$, $f_4 = 1$, $f_3 = f_2 = f_1 = 0$. By Lemma 3, there is a hamiltonian cycle C in A^4. In C we can choose an edge with connection with a faultless subgraph A^i, $i \leq 3$. Through this connection we can extend C into $A^1 \cup A^2 \cup A^3$ and obtain cycles of length from 13 to 47. Suppose for a moment that one faulty vertex $w \in A^5$ is healthy. Then by Lemma 3, there is a hamiltonian cycle C_1 going through w. By removing w from C_1 we obtain the path P_1 going from u to v (two neighbors of w). There is at most one faulty vertex outside A^5. Similarly as in Lemma 3 we can show that u and v can be connected by a path P_2 going through all subgraphs A^1,...,A^4 except one without faults, say A^1. The path P_2 goes through A^2 by hamiltonian path and, by Lemma 4, we can assume it goes by an edge e connected with an edge $f \in A^1$. Combining paths P_1, P_2 and the edge f we can build a cycle of length $48-4+2=46$. Now adding cycles in A^1 going through f we can make cycles up to the maximal length 56.

Case 2. $f_5 = 2$, $f_4 = 2$, $f_3 = f_2 = f_1 = 0$. First, by Lemma 4(3), in A^5 there is a hamiltonian cycle which has connection with an edge f in A^1. It is easy to observe that in A^1 we can find a hamiltonian cycle C which goes through f, 3 separate edges of color 4, and 2 separate edges of each of the colors 2 and 3. Hence, C can be connected with healthy edges in all A^2, A^3, A^4, and A^5. Next we add an edge in A^2, and A^3, the hamiltonian cycle in A^4 and A^5, and we obtain a cycle of length 36. We can extend this cycle by enlarging cycles in subgraphs A^2 and A^3.

Case 3.$f_5 \leq 2$ and $f_4 \leq 1$. Can be proven similarly as Case 2. In A^1 we can find a hamiltonian cycle C which can be connected with healthy edges in all other subgraphs. □

Theorem 9 *Alternating group graphs AG_n, $n \geq 4$, are $(2n-6)$-fault-tolerant pancyclic. That is, if the number of faults $|F| \leq 2n-6$, then it contains a cycle of every length ℓ from 3 to $n!/2 - |F|$.*

Proof. We shall use induction on n. The cases for $n = 4$ and $n = 5$ follow from Lemma 1 and Lemma 8. For $n \geq 6$ let us divide AG_n into subgraphs A^1,...,A^n. We may assume that the sequence f_1,...,f_n is nondecreasing. Note that three smallest f_1, f_2, $f_3 \leq 1$ and $f_4 \leq 2$. Moreover, if $f_4 = 2$ then $f_1 = f_2 = f_3 = 0$.

Case 1. $f_n = 2n-7$. In this case $f_{n-1} = 1$ and $f_i = 0$ for $1 \leq i \leq n-2$. By Lemma 7, in subgraphs A^1,...,A^{n-1} one can build cycles of every length from 3 to $(n-1)(n-1)!/2-1$. Suppose for a moment that one faulty vertex $w \in A^n \cap F$ is healthy. Then by induction hypothesis, there is a hamiltonian cycle C_1 going through w. By removing w from C_1 we obtain the hamiltonian path P_1 going from u to v (two neighbors of w). There is at most one faulty vertex outside A^n, so similarly as in Lemma 3 we can show that u and v may be connected by a path P_2 going through all subgraphs A^1,...,A^{n-1} except one, say A^1. The path P_2 goes through A^2 by hamiltonian path and, by Lemma 4(2), we can assume it goes by an e edge connected with an edge $f \in A^1$. Combining paths P_1, P_2 and the edge f we can build a cycle of length $(n-1)(n-1)!/2-(2n-6)+2$. Now adding cycles in A^1 going through f we can make cycles up to the maximal length $n!/2-(2n-6)$.

Case 2. $f_n \leq 2n-8$ and $f_i \leq n-4$ for $i \leq n-1$. Short cycles from 3 to $h_1+h_2+h_3+h_4$ we build in $A^1 \cup A^2 \cup A^3 \cup A^4$. If $h_4 = 2$ (then $h_1 = h_2 = h_3 = 0$), we first build cycles in $A^1 \cup A^2 \cup A^3$, next we find hamiltonian cycle in A^4 which can be extended into $A^1 \cup A^2 \cup A^3$. By induction hypothesis, there is hamiltonian cycle C_1 in A^n. The length of C_1 is $|C_1| = (n-1)!/2 - f_n$. We can choose $\lfloor((n-1)!/2 - f_n)/2\rfloor$ separate edges in C_1 and there are only $2n-6-f_n$ faulty vertices outside A^n. Hence, we can find an edge e in C_1 with healthy external edges going to two different subgraphs different from A^1, say A^a and A^b, $a \neq 1 \neq b$. Similarly as in Lemma 3 we can show that C_1 can be extended by a path P_2 going through three subgraphs A^a, A^b and A^2 (if $a = 2$ or $b = 2$ then P_2 goes only by 2 subgraphs). The path P_2 goes through A^2 by hamiltonian path and, by Lemma 4, we can assume that it goes by an edge e connected with an edge $f \in A^1$. Moreover f is in a healthy subgraph of A^1. Extending C_1 by P_2 we can build a cycle C_2 of length at most $h_n+h_a+h_b+h_2 \leq h_1+h_2+h_3+h_4$. By Lemma 4(2), the part of P_2 going through A^2 can be changed by a path which omits one vertex. Hence we have also a cycle C_2' of length $|C_2'| = |C_2|-1$ going through an edge connected with A^1. C_2 or C_2' can be extended into A_1 and we get cycles of lengths from $h_n+h_a+h_b+h_2+1$ to $h_n+h_a+h_b+h_2+h_1$. To add next subgraph A^i we require that the path P_2 (and P_2') goes through one more subgraph. In this way we add one by one the rest of the subgraphs.

Case 3. $f_n = f_{n-1} = n-3$. In this case $f_i = 0$ for every $1 \leq i \leq n-2$. Short cycles from 3 to $(n-2)(n-1)!/2$ we build in $A^1 \cup ... \cup A^{n-2}$. By induction hypothesis, there are hamiltonian cycles C_n and C_{n-1} in A^n and A^{n-1} respectively. Each of this cycles can be connected with 3 subgraphs. Indeed, they are of length $(n-1)!/2 - n + 3$ and there are at most $(n-2)!$ edges in one color. Thus we can connect C_n and C_{n-1} with edges e_n and e_{n-1} in two different subgraphs, say A^1 and A^{n-2}. Taking C_n and cycles in $A^1 \cup ... \cup A^{n-2}$ (by Lemma 7, we may assume that these cycles go by e_n) we can build cycles of the length up to $(n-1)(n-1)!/2 - (n-3)$.

In order to build the longest cycles. We take in A^2 two neighboring edges x and y one with connection with an edge $x' \in A^1$ and the other with connection with $y' \in A^3$. In A^1 there is a hamiltonian cycle going through e_n and x'. In $A^3 \cup ... \cup A^{n-2}$ there is hamiltonian cycle going through y' and e_{n-1}. In A^2 there are cycles on any length from 3 to $(n-1)!/2$ going through x and y. □

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