Fault-tolerant pancyclicity in alternating group graphs

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Abstract

In [Z.-J. Xue, S.-Y Liu, An optimal result on fault-tolerant cycle-embedding in alternating group graphs, Inform. Process. Lett. 109 (2009) 1197–1201] the authors claim that every alternating group graph $AG_n$ is $(2n - 6)$ fault tolerant pancyclic. Which means that if the number of faults $|F| \leq 2n - 6$, then $AG_n - F$ contains cycles of every length from 3 to $n!/2 - |F|$. Their proof is not complete. They left a few important things unexplained. In this paper we fulfill these gaps and present another proof that $AG_n$ is $(2n - 6)$-fault-tolerant pancyclic.

Key words: hamiltonian cycle, pancyclicity, alternating group graph, fault tolerance

1 Introduction

An alternating group graph $AG_n$, $n \geq 3$ has vertices labeled by even permutations of the set \{1, ..., n\}. Two vertices $p$ and $q$ are neighbors if one of them is obtained from the other by rotating three symbols: the first, second, and $i$-th, for some $i \geq 3$. There are $n!/2$ vertices in $AG_n$. The graph $AG_3$ has three vertices 123, 231, and 312, every two are connected. $AG_4$ is presented in Fig 1. By $F$ we shall denote the set of faulty vertices. In [2] the authors claim that $AG_n$ is $(2n - 6)$ fault tolerant pancyclic. Which means that if the number of faults $|F| \leq 2n - 6$, then $AG_n - F$ contains cycles of every length from 3 to $n!/2 - |F|$. Their proof, by induction, is not complete. They left a few important things un-
explained. When proving induction step they decompose $AG_n$ into subgraphs $A^1, \ldots, A^n$. By induction hypothesis, shortest cycles are in these subgraphs. To obtain longer cycles they take a cycle $C$ already build and extend it into the next subgraphs using so called 4-cycle structures. If the cycle $C$ is contained in one subgraph $A^i$ then it is easy to see that $C$ can be extended into a new subgraph $A^j$. But the authors do not explain how to find such expanding structure if $C$ goes through more than one subgraph and there are only few subgraphs unvisited. Moreover they do not explain why there are cycles of every length $\ell \geq 3$ going through 4-cycle structure in $A^j$.

In this paper we fulfill the gaps and present another proof that $AG_n, n \geq 4$, is $(2n - 6)$-fault-tolerant pancyclic. We shall prove the theorem by induction. For $n = 4$ the theorem was proven in [1].

**Lemma 1** (see [1]) $AG_4$ is 2-fault tolerant pancyclic.

For $n \geq 5$, we shall decompose $AG_n$ into subgraphs and we show that shortest cycles are constructed in subgraphs with smallest number of faults. In order to build longer cycles we first find a cycle in the subgraphs with many faults and extend it to the rest of the graph. In [2] the authors noted that the bound $2n - 6$ is optimal, because with $2n - 5$ faults we can remove all but one neighbors of a vertex $v$, so no hamiltonian cycle is possible.

## 2 Alternating group graph

An alternating group graph $AG_n, n \geq 3$ has vertices labeled by even permutations of the set $\{1, \ldots, n\}$. The permutation $p = (p_1, \ldots, p_n)$ is even if it contains the even number of inversions. The inversion is a pair of numbers $i, j, 1 \leq i < j \leq n$ such that $p_i > p_j$. For every $i, 3 \leq i \leq n$, let $g_i^+$ be the permutation which rotates symbols in positions 1, 2, and $i$ from left to right; and $g_i^-$ be the permutation which rotates these symbols from right to left. Two vertices $p$ and $q$ are connected by an edge if and only if $q = pg_i^+$ or $q = pg_i^-$ for some $i \geq 3$. Observe that if $q = pg_i^+$ then $p = qg_i^-$. There are $n!/2$ vertices in $AG_n$, and each vertex is connected with $2n - 4$ neighbors. The graph $AG_n$ can be divided into subgraphs $A^1, \ldots, A^n$, each $A^i$ contains vertices with $i$ on the last symbol. The subgraph $A^1$ is isomorphic with $AG_{n-1}$. We can also divide $AG_n$ according to other position, say $k$, for some $3 \leq k \leq n - 1$. Then $A^i$ contains vertices with $i$ on the $k$-th position. Note that every two vertices $u$ and $v$ must differ in some symbol $k \geq 3$ and we can decompose $AG_n$ in such a way that $u$ and $v$ are in different subgraphs, and we can always assume that faulty vertices are not in one subgraph. On the other hand we can also divide $AG_n, n \geq 4$, in such a way that two ends of an edge are in one subgraph. This is because they differ only in one position $i \geq 3$. Every vertex $u \in A^i$ is connected with exactly two vertices $u'$ and $u''$ which are in two different subgraphs outside $A^i$. We will call the edges $(u, u')$ and
(u, u'') external edges. Other edges we shall call internal. For each internal edge (u, v) ∈ A^i with u = (kj...i) and v = (jk'i...j) there exist adjacent vertices s = (ik...j) and t = (k'i...j) both in A^j which form the 4-cycle (u, s, t, v). We shall say that the edge (u, v) is of color j or that it is connected (by a 4-cycle) with the edge (s, t) in A^j. If a subgraph A^i is of dimension 4 and is isomorphic to AG_4 (see Fig. 1) then there are 4 colors, the edges of each color form a cycle of length 6. For example, the cycle 1234, 4132, 1342, 2143, 1423, 3124 contains edges of color 1. If a subgraph A^i is of dimension 5 then it can be divided into 5 subgraphs A_1^i, A_2^i, A_3^i, A_4^i, A_5^i (according to the 5-th position). Each of A_j^i is isomorphic to AG_5, and contains 4 colors (all colors except i and j), and edges in each color form a cycle of length 6. Similarly for higher dimensions. AG_n can be divided into n subgraphs A^1,...,A^n according to the last position. Each A^i can be divided into (n - 1) subgraphs A_1^i,...,A_{n-1}^i according to the last by one position and so on. But the color of the edge depends only on the first two symbols and is the same in each subgraph. Moreover if an edge (u, v) is in the subgraph A_j^i and is connected with the edge (u', v') in A^i, then the edge (u', v') is in the subgraph A_{ik}^j. There are (n - 2)! external edges joining two different subgraphs A^i and A^j. If x and y are two vertices in a subgraph, say A^1, then it is easy to see that we can choose external edges (x, x') and (y, y') in such a way that x' and y' are in two different subgraphs. Moreover if x and y are neighbors and A^j is some other subgraph j ≥ 2 then we can choose x' and y' to be not in A^j. This is because if x is connected by external edges with x' and x'' in two different subgraphs and y is connected with y' and y'' also in two different subgraphs, and x' and y' are in one subgraph then (x, y, y', x') form a 4-cycle structure and there is only one 4-cycle structure for the edge (x, y). By F we shall denote the set of faulty vertices; f_i = |A^i ∩ F| denotes the number of faulty vertices in A^i, and h_i = |A^i - F| denotes the number of healthy vertices in A^i.

**Lemma 2** (see [2]) Let A^1,..., A^k be arbitrary string of subgraphs from the decomposition of AG_n, 2 ≤ k ≤ n. Each A^i − F is hamiltonian connected and each A^i is connected with A^{i+1} by at least 3 healthy edges. Then for every s ∈ A^1 and t ∈ A^k there is hamiltonian path connecting s and t in A^1 ∪ ... ∪ A^k.

We say that a graph is hamiltonian connected if for every two vertices u and v, there is hamiltonian path from u to v.

**Lemma 3** AG_n, n ≥ 3, is hamiltonian connected if |F| ≤ n - 3.

**Proof.** By induction on n. It is easy to check that the lemma is valid for n = 3 or n = 4. For n ≥ 5 let us divide AG_n into subgraphs A^1,...,A^n. We can assume that faulty nodes are not in one subgraph and f_i ≤ n - 4 = (n - 1) - 3 for all i. Suppose first that u and v are in the same subgraph, say in A^1. By induction hypothesis, there is hamiltonian path P_i in A^1 connecting u and v. The length of P_i is |P_i| = (n - 1)!/2 -
Consider decomposition of $AG_{n+1}$ into subgraphs $A^1,\ldots,A^{n+1}$ of dimension $n$ and let $A^i$ and $A^k$ be two of them, and let $u,v$ be arbitrary two vertices in $A^i$ ($u$ and $v$ may form an edge).

1. If $n = 4$ and $f_1 = 1$ then there is a hamiltonian path in $A^i$ going from $u$ to $v$ through an edge of color $k$.

2. If $n \geq 5$ and $f_i, f_k \leq 1$. Then there is a hamiltonian path $P$ in $A^i$ from $u$ to $v$ which goes through an edge $e$ which is connected with an edge $e' \in A^k$ standing in a faultless subgraph $A^\ell_m$ of $A^\ell$. There is also a path $P'$ of length $|P'| = |P| - 1$ with the same property.

3. If $n \geq 4$ and $f_1 = 2$ then there is a hamiltonian cycle in $A^i$ going through an edge of color $k$.

**Proof:** (1) can be easily proven by examining all cases. To prove (2) consider decomposition of $A^i$ into subgraphs $A^i_1,\ldots,A^i_m$. We can assume that $u$ and $v$ are in different subgraphs. Similarly as in Lemma 3 we can show that $u$ and $v$ can be connected by a path going through all subgraphs by a hamiltonian path. At least 4 subgraphs contain color $k$. Thus we can choose a subgraph $A^i_m$ with color $k$, such that the subgraph $A^i_m$ and $A^k_m$ are free of faults. By (1) or induction hypothesis, we can assume that the path goes by color $k$ in $A^i_m$. In order to obtain the path $P'$, we omit one vertex in some faultless subgraph other than $A^i_m$. This is possible because $AG_n$ without faults is panconnected [1].

(3) Proof by induction on $n$. It is easy to check that the lemma is valid for $n = 4$. For $n \geq 5$ let us divide $A^i$ into subgraphs $A^i_1,\ldots,A^i_m$. We may assume that $A^i_1$ contains at least one fault. By induction hypothesis, there is hamiltonian cycle $C$ in $A^i_1$. There is at most one fault outside $A^i_1$, so we can find an edge $(x,y)$ in $C$ with healthy external edges going to two different subgraphs. Similarly as in the proof of (2) we can extend $C$ by a path $P$ going through the rest of $A^i$ and visiting color $k$.

**Lemma 5** For any two edges $e,f \in AG_n$, $n \geq 3$, there exists a hamiltonian cycle going through $e$ and $f$.

**Proof.** By induction on $n$. It is easy to see that the lemma is true for $n = 3$ or $n = 4$. Let $n \geq 5$. Decompose $AG_n$ into subgraphs $A^1,\ldots,A^n$. We may assume that both ends of $e$ (and $f$) are
in one subgraph (e and f may be in different subgraphs). This is because
there is a position $j \geq 3$ such that neither e nor f differ in position $j$.

Case 1. Edges $e$ and $f$ are in one subgraph, say in $A^1$. By induction hypothesis, there is a hamiltonian cycle $C$ going through $e$ and $f$. There is a third edge in $C$ which connects $C$ with another subgraph and further with the rest of $AG_n$.

Case 2. Edges $e$ and $f$ are in different subgraphs, say in $A^1$ and $f$ in $A^n$. For every $i$, $1 \leq i \leq n$; we choose two edges $e_i$ and $f_i$ in $A_i$ such that $f_i$ and $e_{i+1}$ form a 4-cycle; and $e = e_1$, $f_n = f$. By induction hypothesis, there is a hamiltonian cycle in $A_i$ going through $e_i$ and $f_i$. All these cycles can be connected in one hamiltonian cycle.

We say that a graph $G$ is $k$-edge-pancyclic if for every edge $e$ there is a cycle going through $e$ of every length from $k$ to $|G|$.

**Lemma 6**

(1) $AG_n$ with $|F| = 0$ and $n \geq 3$ is 3-edge-pancyclic.

(2) $AG_4$ with $|F| = 1$ is 5-edge-pancyclic.

(3) $AG_n$ with $|F| = 1$ and $n \geq 5$, is 4-edge-pancyclic. Moreover if an edge $e$ is in faultless subgraph, then there is also a cycle of length $3$ going through $e$.

Proof (1) follows from the symmetry of $AG_n$. (2) can be easily proven looking through all cases. To prove (3) let us decompose $AG_n$ into subgraphs $A^1,...,A^n$. We can assume that both ends of $e$ are in one subgraph, say $A^1$.

Case 1. The faulty vertex is also in $A^1$. The edge $e$ is connected by 4-cycle with an edge outside $A^1$. By induction hypothesis, or by (2), there is a cycle going through $e$ of every length from 5 to $(n - 1)!/2 - 1$. Hence there is a cycle $C$ of length $(n - 1)!/2 - 1$ and a cycle $C'$ of length $(n - 1)!/2 - 2$ going through $e$. We choose an edge $f$ in $C$ (and $f'$ in $C'$). The edge $f$ is connected with an edge $e_2$ in another subgraph, say $A^2$ ($f'$ is connected with an edge $e_2'$ in $A^2$). From $C$ and $e_2$ one can build a cycle of length $|C| + 2$. Similarly using $C'$ one can build a cycle of length $|C| + 1$. There is a cycle in $A^2$ going through $e_2$ of every length from 3 to $(n - 1)!/2$. Joining this cycle with $C$ we obtain cycles of length from $|C| + 3$ to $|C| + (n - 1)!/2$. We proceed in a similar manner in order to extend these cycles into the next subgraphs. First we get in $A^2$ an edge $f_2$ which has a connection with $A^3$. By Lemma 5, there is a hamiltonian cycle $C_1$ in $A^2$ going through $e_2$ and an edge $f_2$ (and $C_2'$ going through $e_2'$ and $f_2$). Joining $C$ and $C_1$ we obtain the cycle of length $|C| + (n - 1)!/2$ going by $f_2$. Joining $C'$ with $C_1$ we obtain the cycle of length $|C| + (n - 1)!/2 - 1$ going by an $f_2$. Similarly as before we can extend this cycles to the cycles of every length from $|C| + (n - 1)!/2 + 1$ to $|C| + (n - 1)!$ and further to the next subgraphs.

Case 2. The faulty vertex is in another subgraph, say in $A^n$. By (1), there is a cycle going through $e$ of every length from 3 to $(n - 1)!/2$. Similarly as in Case 1, we can extend these
cycles into subgraphs $A^2, \ldots, A^{n-1}$. To obtain the longest cycles, we first get a hamiltonian cycle $C$ in $A^1$ which goes by $e$ end has connection with $A^n$, this is possible by Lemma 5. In $A^n$ we add the hamiltonian cycle. Next we choose the edge in $C$ with connection with another subgraph and through this connection we extend cycles into the rest of subgraphs.

\[\square\]

**Lemma 7** (1) Let $A_1, \ldots, A_k$ be a sequence of subgraphs of dimension $n$, $n \geq 4$ without faults; $e$ be an edge in $A^1$. Then there is a cycle in $A_1 \cup \ldots \cup A_k$ going through $e$ of every length $\ell$, $3 \leq \ell \leq k \cdot n!/2$.

(2) Additionally let $g$ be an arbitrary edge in $A^k$. Then there is a hamiltonian cycle in $A_1 \cup \ldots \cup A_k$ going through $e$ and $g$.

(3) Let $A_1, \ldots, A_k$ be a sequence of subgraphs of dimension $n$, $n \geq 5$ with at most one fault each; $e$ be an edge in $A^1$. Then there is a cycle in $A_1 \cup \ldots \cup A_k$ going through $e$ of every length $\ell \geq 3$.

**Proof:** (1) Since $AG_n$ is pancyclic there is cycle in $A^1$ of length $\ell$ for $3 \leq \ell \leq n!/2$. By symmetry, we can assume that every of this paths goes through $e$. By Lemma 4, there is a hamiltonian cycle $C_1$ in $A^1$ going through $e$ and an edge $f$ connected with $A^2$ and there is a cycle $C'_1$ of length $|C'_1| = 1$ going through $e$ and an edge $f'$ connected with $A^2$. Similarly as in the proof of Lemma 6 we can show that these cycles can be extended into every length greater than $|C|$.

(2) When we construct the hamiltonian cycle going through all subgraphs, then by Lemma 5, we can choose the cycle in the last subgraph $A^k$ in such a way that it goes through the edge $g$.

(3) The proof is similar to that of (1). By Lemma 4(2), in $A^1$ there is a hamiltonian cycle $C$ and a cycle $C'$ shorter by one which goes by $e$ and can be extended into $A^2$. We use Lemma 4(2) in order to extend cycles into next subgraphs.

\[\square\]

4 Main result

**Lemma 8** $AG_5$, is 4-fault-tolerant pancyclic. That is, if the number of faults $|F| \leq 4$, then it contains a cycle of every length $\ell$ from 3 to $60 - |F|$.

We decompose $AG_5$ into subgraphs $A_1, \ldots, A^5$. We can assume that the sequence $f_1, \ldots, f_5$ is nondecreasing. Since $|F| = 4$ we have $f_1 = 0$ and $f_3 \leq 1$. Shortest cycles of lengths from 3 to $h_1 + h_2 + h_3 \geq 34$. We build in $A^1 \cup A^2 \cup A^3$. First in $A_1$, then in $A_1 \cup A_2$ and at the end we build a hamiltonian cycle in $A_1$ and extend it into $A_2$ and $A_3$. Next we build the longest cycles.

Case 1. $f_5 = 3$, $f_4 = 1$, $f_3 = f_2 = f_1 = 0$. By Lemma 3, there is a hamiltonian cycle $C$ in $A^4$. In $C$ we can choose an edge with connection with a faultless subgraph $A^i$, $i \leq 3$. Through this connection we can extend $C$ into $A^1 \cup A^2 \cup A^3$ and obtain cycles of length from 13 to 47. Suppose for a moment that one faulty
vertex $w \in A^5$ is healthy. Then by Lemma 3, there is a hamiltonian cycle $C_1$ going through $w$. By removing $w$ from $C_1$ we obtain the path $P_1$ going from $u$ to $v$ (two neighbors of $w$). There is at most one faulty vertex outside $A^3$. Similarly as in Lemma 3 we can show that $u$ and $v$ can be connected by a path $P_2$ going through all subgraphs $A^1, \ldots, A^4$ except one without faults, say $A^1$. The path $P_2$ goes through $A^2$ by hamiltonian path and, by Lemma 4, we can assume it goes by an edge $e$ connected with an edge $f \in A^1$. Combining paths $P_1$, $P_2$ and the edge $f$ we can build a cycle of length $48 - 4 + 2 = 46$. Now adding cycles in $A^1$ going through $f$ we can make cycles up to the maximal length 56.

Case 2. $f_3 = 2$, $f_4 = 2$, $f_5 = f_2 = f_1 = 0$. First, by Lemma 4(3), in $A^5$ there is a hamiltonian cycle which has connection with an edge $f$ in $A^1$. It is easy to observe that in $A^1$ we can find a hamiltonian cycle $C$ which goes through $f$, 3 separate edges of color 4, and 2 separate edges of each of the colors 2 and 3. Hence, $C$ can be connected with healthy edges in all $A^2$, $A^3$, $A^4$, and $A^5$. Next we add an edge in $A^2$, and $A^3$, the hamiltonian cycle in $A^4$ and $A^5$, and we obtain a cycle of length 36. We can extend this cycle by enlarging cycles in subgraphs $A^2$ and $A^3$.

Case 3. $f_5 \leq 2$ and $f_4 \leq 1$. Can be proven similarly as Case 2. In $A^1$ we can find a hamiltonian cycle $C$ which can be connected with healthy edges in all other subgraphs.

\[ \text{Theorem 9} \quad \text{Alternating group graphs} \]

$AG_n, n \geq 4$, are $(2n - 6)$-fault-tolerant pancyclic. That is, if the number of faults $|F| \leq 2n - 6$, then it contains a cycle of every length $\ell$ from 3 to $n!/2 - |F|$.

Proof. We shall use induction on $n$. The cases for $n = 4$ and $n = 5$ follow from Lemma 1 and Lemma 8. For $n \geq 6$ let us divide $AG_n$ into subgraphs $A^1, \ldots, A^n$. We may assume that the sequence $f_1, \ldots, f_n$ is nondecreasing. Note that three smallest $f_1$, $f_2$, $f_3 \leq 1$ and $f_1 \leq 2$. Moreover, if $f_4 = 2$ then $f_1 = f_2 = f_3 = 0$.

Case 1. $f_n = 2n - 7$. In this case $f_{n-1} = 1$ and $f_i = 0$ for $1 \leq i \leq n - 2$. By Lemma 7, in subgraphs $A^1, \ldots, A^{n-1}$ one can build cycles of every length from 3 to $(n - 1)(n - 1)!/2 - 1$. Suppose for a moment that one faulty vertex $w \in A^n \cap F$ is healthy. Then by induction hypothesis, there is a hamiltonian cycle $C_1$ going through $w$. By removing $w$ from $C_1$ we obtain the hamiltonian path $P_1$ going from $u$ to $v$ (two neighbors of $w$). There is at most one faulty vertex outside $A^n$, so similarly as in Lemma 3 we can show that $u$ and $v$ may be connected by a path $P_2$ going through all subgraphs $A^1, \ldots, A^{n-1}$ except one, say $A^1$. The path $P_2$ goes through $A^2$ by hamiltonian path and, by Lemma 4(2), we can assume it goes by an edge connected with an edge $f \in A^1$. Combining paths $P_1$, $P_2$ and the edge $f$ we can build a cycle of length $(n - 1)(n - 1)!/2 - (2n - 6) + 2$. Now adding cycles in $A^1$ going through $f$ we can make cycles up to the maximal length $n!/2 - (2n - 6)$. 

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Case 2. $f_n \leq 2n - 8$ and $f_i \leq n - 4$ for $i \leq n - 1$. Short cycles from 3 to $h_1 + h_2 + h_3 + h_4$ we build in $A^1 \cup A^2 \cup A^3 \cup A^4$. If $h_4 = 2$ (then $h_1 = h_2 = h_3 = 0$), we first build cycles in $A^1 \cup A^2 \cup A^3$, next we find hamiltonian cycle in $A^4$ which can be extended into $A^1 \cup A^2 \cup A^3$. By induction hypothesis, there is hamiltonian cycle $C_1$ in $A^n$. The length of $C_1$ is $|C_1| = (n - 1)!/2 - f_n$. We can choose $[((n - 1)!/2 - f_n)/2]$ separate edges in $C_1$ and there are only $2n - 6 - f_n$ faulty vertices outside $A^n$. Hence, we can find an edge $e$ in $C_1$ with healthy external edges going to two different subgraphs different from $A^1$, say $A^a$ and $A^b$, $a \neq 1 \neq b$. Similarly as in Lemma 3 we can show that $C_1$ can be extended by a path $P_2$ going through three subgraphs $A^a$, $A^b$ and $A^2$ (if $a = 2$ or $b = 2$ then $P_2$ goes only by 2 subgraphs). The path $P_2$ goes through $A^2$ by hamiltonian path and, by Lemma 4, we can assume that it goes by an edge $e$ connected with an edge $f \in A^1$. Moreover $f$ is in a healthy subgraph of $A^1$. Extending $C_1$ by $P_2$ we can build a cycle $C_2$ of length at most $h_n + h_a + h_b + h_2 \leq h_1 + h_2 + h_3 + h_4$. By Lemma 4(2), the part of $P_2$ going through $A^2$ can be changed by a path which omits one vertex. Hence we have also a cycle $C'_2$ of length $|C'_2| = |C_2| - 1$ going through an edge connected with $A^1$. $C_2$ or $C'_2$ can be extended into $A_1$ and we get cycles of lengths from $h_n + h_a + h_b + h_2 + 1$ to $h_n + h_a + h_b + h_2 + h_1$. To add next subgraph $A^1$ we require that the path $P_2$ (and $P'_2$) goes through one more subgraph. In this way we add one by one the rest of the subgraphs.

Case 3. $f_n = f_{n-1} = n-3$. In this case $f_i = 0$ for every $1 \leq i \leq n - 2$. Short cycles from 3 to $(n - 2)(n - 1)!/2$ we build in $A^1 \cup \ldots \cup A^{n-2}$. By induction hypothesis, there are hamiltonian cycles $C_n$ and $C_{n-1}$ in $A^n$ and $A^{n-1}$ respectively. Each of this cycles can be connected with 3 subgraphs. Indeed, they are of length $(n - 1)!/2 - n + 3$ and there are at most $(n - 2)!$ edges in one color. Thus we can connect $C_n$ and $C_{n-1}$ with edges $e_n$ and $e_{n-1}$ in two different subgraphs, say $A^1$ and $A^{n-2}$. Taking $C_n$ and cycles in $A^1 \cup \ldots \cup A^{n-2}$ (by Lemma 7, we may assume that these cycles go by $e_n$) we can build cycles of the length up to $(n - 1)(n - 1)!/2 - (n - 3)$.

In order to build the longest cycles. We take in $A^2$ two neighboring edges $x$ and $y$ one with connection with an edge $x' \in A^1$ and the other with connection with $y' \in A^3$. In $A^1$ there is a hamiltonian cycle going through $e_n$ and $x'$. In $A^3 \cup \ldots \cup A^{n-2}$ there is hamiltonian cycle going through $y'$ and $e_{n-1}$. In $A^2$ there are cycles on any length from 3 to $(n - 1)!/2$ going through $x$ and $y$. \hfill \Box

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