The Gonzalez and Sahni Algorithm for $O|pmtn|C_{max}$

- *n machines* $M_1, \ldots, M_n$
- *n jobs* $J_1, \ldots, J_n$, each job requiring processing on every machine, in any order.
- $p_{ij} \geq 0$ processing time of job J_j on machine M_i
- each machine can process at most one job at a time
- no job can be processed simultaneously on two or more machines.

10	20	0
0	10	10
10	0	5

Example: processing times

Let

$$R_i = \sum_{j=1}^{n} p_{ij} (i = 1, \ldots, n)$$

$$C_j = \sum_{i=1}^{m} p_{ij} (j = 1, \ldots, n)$$

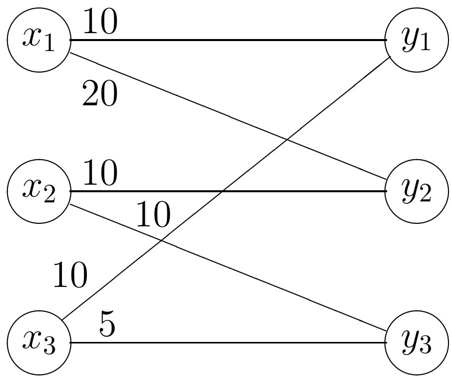
an obvious lower bound on the optimal solution value is

$$\alpha = \max(\max_i \{R_i\}, \max_j \{C_j\}) \quad (1)$$

0) Define a weighted bipartite graph $G = (X \cup Y, E)$ in which X has a vertex x_i for each machine M_i, Y has a vertex y_j for each job J_j and $E = \{(x_i, y_j) : t_{ij} > 0, 1 \leq i \leq n, 1 \leq j \leq n\}$. The weight of an edge $(x_i, y_j) \in E$ is the processing time p_{ij}.

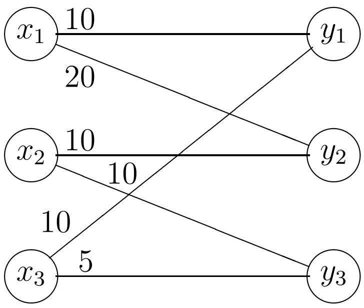
10	20	0
0	10	10
10	0	5

processing times

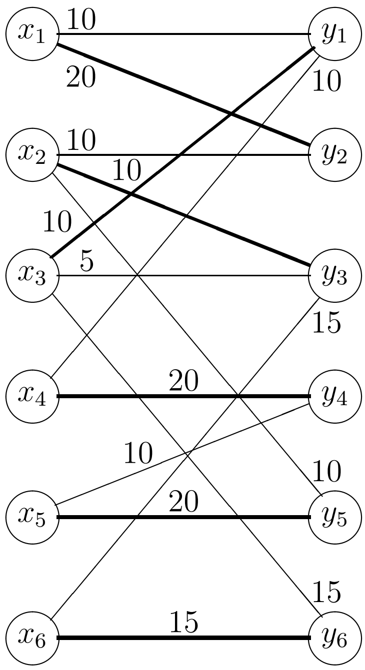


graph

1) Add n vertices $x_{n+1}, \ldots, x_{2n}$ to X and n vertices $y_{n+1}, \ldots, y_{2n}$ to Y.
Add the following edges to E:
(i) an edge (x_i, y_{n+i}) for each i such that $\alpha - R_i > 0$, with weight $\alpha - R_i$ (see (1));
(ii) an edge (x_{n+j}, y_j) for each j such that $\alpha - C_j > 0$, with weight $\alpha - C_j$;
(iii) a number of edges connecting $x_{n+1}, \ldots, x_{2n}$ to $y_{n+1}, \ldots, y_{2n}$, with weights such that the weight sum of the edges incident to each vertex is equal to α.



(a) Original graph



(b) Extended graph

2) Find a complete matching $\mathcal{M}$ (i.e., a set of n edges with no vertex in common), and let μ be the minimum weight of an edge in $\mathcal{M}$.
(For the example of the figure, we could find the edges drawn in thick lines with $\mu = 10$)
Define a processing phase scheduling, for each edge $(x_i, y_j) \in \mathcal{M}$, job J_j on machine M_i for μ time units (but if $j > n$ then machine M_i is idle in that interval; if $i > n$ then job J_j is not processed in that interval).

3) Decrease the weight of each edge in $\mathcal{M}$ by μ and remove all edges with zero weight. If the edge set is empty, terminate; otherwise, go to 2.