The Lighthouse Problem*

Navigating by Lighthouses in Geometric Domains

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Abstract

We study the computational properties of placing a minimum number of lighthouses in different geometric domains and under different notions of visibility, enabling a vehicle placed anywhere in the domain to navigate to a given specific target. This problem shares common elements with the art gallery problem in that the whole domain must be covered with as few lighthouses as possible. Our main result is an algorithm that places a minimum set of *strip lighthouses* in a simple rectilinear polygon. These correspond to sliding cameras in art gallery vernacular.

> Unfortunately, at the time of presentation of these results, the authors encountered a counterexample to Lemma 4, thus invalidating the results claimed up to and including Theorem 7.

1 Introduction

We consider a problem lying in the intersection of *routing amongst obstacles* and *the art gallery problem*. Our problem is that of placing as few landmarks in a domain as possible such that a vehicle (being a ship, an airplane or a drone) can safely navigate the domain to reach a specified target. It is related to the routing problem since the vehicle should be guaranteed to avoid the obstacles while following a simple routing protocol to reach the target. At the same time, the landmarks must "cover" the whole domain to ensure that the vehicle can begin to navigate from any point in the domain, thus connecting our problem to the art gallery problem [18].

Beacon-based direct-visibility routing was used in the early days of aviation to guide (badly equipped) airplanes. The airplane would fly in the direction of the beacon until a beacon closer to the target could be seen from the plane. The plane would then continue in the direction of the new beacon, hopping from beacon to beacon until it reached the target [19]. Minimizing the number of beacons to place in a domain was therefore important to make beacon-based direct-visibility routing practically feasible. Herein, we consider the two-dimensional variant of this problem and to emphasize this, we use the concepts of *lighthouses* and *ships* rather than beacons and airplanes or drones.

Our navigation protocol for the ships is very simple but places certain restrictions on the placement of the lighthouses in the domains, sometimes making our placement problem computationally easier than the corresponding art gallery problem. Each lighthouse has an associated identifying number that is transmitted through the lighthouse signal. Thus, each ship can identify the lighthouse it is moving towards. Our *standard navigation protocol* specifies that the ship should move towards the lighthouse with the smallest identifying number that it has currently seen. The target always has identifying number 0 while the other lighthouses should have successively larger numbers as we move away from the target in the domain.

The Lighthouse Problem (LP)

> Given a domain and a *target* t in the domain, determine the minimum number of lighthouses, together with their locations and identifying numbers, ensuring that a ship starting from any position in the domain can travel to t with the *standard navigation protocol*: move towards the lighthouse with the smallest identifying number that is visible.

We can thus identify models of lighthouse problems by specifying different domain, lighthouse, and visibility types. We consider two general variants in this paper, defined by the domain type: in Section 2, we consider the lighthouse problem in grid domains, while in Section 3 — in rectilinear polygons, in both cases with strip lighthouses that define the lighthouse type and the visibility (the SLP problem for short). A *strip lighthouse* is an axis-aligned line segment l that can *guide/attract/is visible to* a point p in the domain, if the perpendicular projection of p onto l is not exterior to the domain. Finally, we also give some basic results for edge lighthouses in rectilinear polygons and for laser lighthouses in grids. A *laser* is a point that can illuminate in exactly one of the four compass directions.

Background. Our lighthouse problem is a variant of the art gallery problem, originally posed by Klee in 1973

*The authors were supported by the grants 2018-04001 (Swedish Research Council) and 2015/17/B/ST6/01887 (National Science Centre, Poland).

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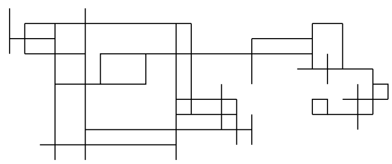

Figure 1: An example of a grid.

as the question of determining the minimum number of guards sufficient to see every point of the interior of a simple polygon; for more details, see O'Rourke [18], Shermer [20], Urrutia [21] — combined with the concept of cooperative guards [13, 15, 22] in which visibility-based connectivity between guards is required. We state this crucial property more formally.

Property 1 *If $\mathcal{L}$ is a solution to LP in some domain, then the visibility graph of $\mathcal{L}$ (under the visibility model considered) is connected.*

The visibility model we consider, immediately relates our lighthouse problem to the Minimum Sliding Cameras (MSC) problem introduced by Katz and Morgenstern [14] and then studied in [3, 4, 5, 6, 7, 8, 9, 12]. For that problem, combinatorial lower and upper bounds on the minimum number of sliding cameras are provided in [5, 12] and it is shown that the MSC problem is NP-hard in polygons with holes [5, 8] but admits a PTAS in simple polygons [3] — but so far NP-hardness in simple polygons still remains an open problem, although there exist linear-time exact algorithms with some additional assumptions either on the input polygon or the solution itself [4, 12].

For this visibility model, we may always assume that the strip lighthouses are maximal (within the domain) and furthermore, Property 1 translates to the fact that the union of strip lighthouses that together allow to navigate within the domain is a connected set.

2 Navigating in Grids

A *grid* is an arrangement of distinct vertical and horizontal closed line segments in the plane, where every two collinear line segments are disjoint and their union is connected; the grid can be thought of as a polygon with holes, representing a region of intersecting very thin corridors, see Figure 1 for an example. So a *strip lighthouse* in a grid is a subsegment of a grid segment and visibility is considered to be perpendicular to its direction along the grid segments intersecting that subsegment. Recall that a strip lighthouse is always assumed to be maximal and therefore we may identify it with the corresponding grid segment. Also, we assume that the target is a complete grid segment, to avoid issues with objects on that grid segment but not on the target.

Observe that for any feasible solution to the LP problem in a grid $\mathbf{G}$, any grid segment must be intersected by at least one lighthouse, so strip lighthouses must constitute a complete cover of $\mathbf{G}$. Since the union of strip lighthouses that together allow to navigate within $\mathbf{G}$ is a connected set, it follows from [15] that the LP problem in $\mathbf{G}$ can be solved by reduction to the minimum cooperative mobile guard set problem in the grid obtained from $\mathbf{G}$ by adding a new grid segment intersecting only the target segment (to force that target segment to be included in the optimal solution; identifying number may be then assigned in a greedy DFS-like manner, starting with 0 for the target segment). On the other hand, the minimum cooperative mobile guard set problem in $\mathbf{G}$ can be solved by reduction to the LP problem in that grid (by taking the best solution over those resulting from checking each of the grid segments as a possible candidate for the target segment). Consequently, as the problem of finding a minimum connected mobile guard set is NP-hard [15], we obtain the following result.

Corollary 2 *The SLP problem in grids is NP-hard.*

Furthermore, since there is one-to-one correspondence between a minimum cooperative mobile guard set in a grid $\mathbf{G}$ and the minimum dominating set of the intersection graph of $\mathbf{G}$, and Guha and Khuller [11] proposed an $O(\log \Delta)$-approximation algorithm for computing the minimum connected dominating set of a graph, where Δ is the maximum degree of that graph, and proved a lower bound of $\Omega(\log \Delta)$ even for bipartite graphs, we may conclude the following corollary.

Corollary 3 *The SLP problem in grids can be approximated with an $O(\log \Delta)$ approximation ratio, where Δ is the maximum number of intersections on a grid segment.*

3 Navigating in Rectilinear Polygons

Our main result of this section is a quadratic time algorithm for computing an optimum set of strip lighthouses in a simple rectilinear polygon. The input to the algorithm is a rectilinear polygon $\mathbf{P}$ and a specified target edge t of $\mathbf{P}$.

We first observe that the target t must be considered to be a strip lighthouse and so included in the optimum set of strip lighthouses for $\mathbf{P}$, with the identifying number 0. Following this observation, consider the *histogram partition* $\mathcal{H}$ of $\mathbf{P}$ with t as the base, see Figure 2(a), for which the dual graph T is a tree rooted at the histogram having t as its base, and with the *windows* of a histogram recursively acting as the *bases* for the relevant histograms corresponding to child nodes in T [16]. Recall that a *histogram* is a rectilinear x- or y-monotone polygon with one boundary chain being a

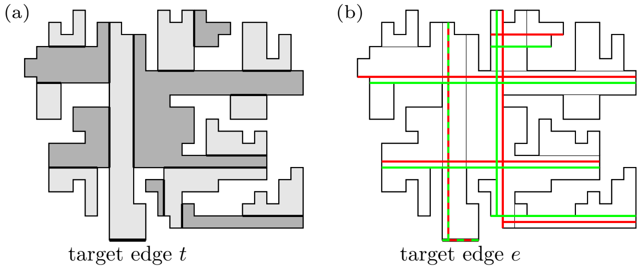

Figure 2: (a) Partition into histograms, starting from the target edge t. (b) An optimal solution (red) to the LP problem and the relevant canonical one (green).

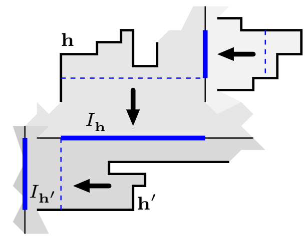

Figure 3: Illustrating the propagation of intervals.

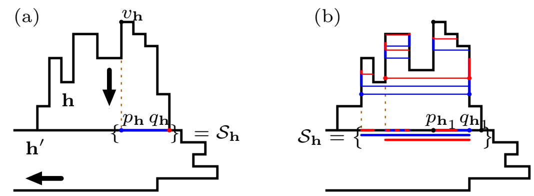

Figure 4: Defining the histogram intervals.

line segment (called the *base*). Now, for each histogram $\mathbf{h}$ in $\mathcal{H}$, we associate the *base direction* of $\mathbf{h}$ to be the direction towards its base, denoted $b_{\mathbf{h}}$, and for a set $\mathcal{L}$ of strip lighthouses in $\mathbf{P}$, we define the *canonical set* with respect to $\mathcal{L}$ as follows. First, we modify the set $\mathcal{L}$ by recursively considering each histogram $\mathbf{h}$ in $\mathcal{H}$ corresponding to a leaf in T. For every strip lighthouse l intersecting $\mathbf{h}$ and being parallel to $b_{\mathbf{h}}$ (assuming l to be maximal), we move l continuously in the base direction as far as possible while not decreasing the visibility region of l. If $\mathbf{h}$ contains several such lighthouses, the movement is done for each lighthouse in sequence in the order of decreasing distance to the base $b_{\mathbf{h}}$. Once this process has been completed for each lighthouse in $\mathbf{h}$ parallel to $b_{\mathbf{h}}$, we remove any lighthouses that coincide, thus reducing the size of the canonical set. Next, we remove the corresponding node from T and repeat the process until all histograms of $\mathcal{H}$ have been considered; see Figure 2(b) for an example. Observe that any strip lighthouse in a canonical set intersecting a histogram and being parallel to the base, is not completely contained in the histogram.

Our idea is then to associate a canonical feasible set of strip lighthouses in $\mathbf{P}$ with a set $\mathcal{S}_{\mathbf{h}}$ of pairs of 0/1-valued intervals on the base $b_{\mathbf{h}}$ of each histogram $\mathbf{h}$ that a maximal strip lighthouse orthogonal to $b_{\mathbf{h}}$ can intersect. For convenience, we identify the first interval in an interval pair as blue and the second one as red, respectively, using $B_{\mathbf{h}}$ and $R_{\mathbf{h}}$ to denote them when no confusion arise; an interval denoted $I_{\mathbf{h}}$ could be either blue or red.

For the purpose of defining the set $\mathcal{S}_{\mathbf{h}}$, we start with a few definitions. Consider two histograms $\mathbf{h}$ and $\mathbf{h}'$, $\mathbf{h}$ being a child of $\mathbf{h}'$ in T, and assume for the definition that the base direction of $\mathbf{h}$ is down and the base direction of $\mathbf{h}'$ is left; see Figure 3. When we henceforth define objects in histograms, we will always make this assumption, thereby avoiding having to define each object also for the other seven possible cases. Let $I_{\mathbf{h}}$ be an interval lying on the base $b_{\mathbf{h}}$ (which is a window of $\mathbf{h}'$). We first project the endpoint of $I_{\mathbf{h}}$ that is closest to the base $b_{\mathbf{h}'}$ vertically onto the opposite horizontal edge of $\mathbf{h}'$ — this gives us two endpoints of a vertical line segment in $\mathbf{h}'$. Next, we project this line segment horizontally onto the base $b_{\mathbf{h}'}$ — this gives us the interval $I_{\mathbf{h}'}$. Following this sequence of two projections, we say that $I_{\mathbf{h}}$ *propagates* from $\mathbf{h}$ to $I_{\mathbf{h}'}$ in $\mathbf{h}'$ in one step, which is denoted by $I_{\mathbf{h}'} = \mathrm{pr}(I_{\mathbf{h}})$. An interval can thus be propagated using a sequence of one-step propagations from a histogram to any ancestral histogram in T; again see Figure 3. Finally, we need to define some special points in a histogram $\mathbf{h}$; see Figure 4(a). First, the point $q_{\mathbf{h}}$ is the rightmost point of $b_{\mathbf{h}}$. Next, we follow the boundary of $\mathbf{h}$ from $q_{\mathbf{h}}$ in counterclockwise order along the xy-monotone staircase until the end of that staircase at vertex $v_{\mathbf{h}}$ is encountered (the vertex $v_{\mathbf{h}}$ is a convex vertex with the adjacent vertical edge below $v_{\mathbf{h}}$ and the adjacent horizontal edge to the right of $v_{\mathbf{h}}$) — projecting a point vertically from $v_{\mathbf{h}}$ onto $b_{\mathbf{h}}$ defines the point $p_{\mathbf{h}}$.

Thereby, the encoding $\mathcal{S}_{\mathbf{h}}$ is defined recursively as follows (see Figure 4(b)):

$$\mathcal{S}_{\mathbf{h}} = \Big\{ \big(\mathrm{pr}(B_{\bar{\mathbf{h}}}), \mathrm{pr}(R_{\bar{\mathbf{h}}})\big) \,|\, (B_{\bar{\mathbf{h}}}, R_{\bar{\mathbf{h}}}) \in \mathcal{S}_{\bar{\mathbf{h}}}, \forall \bar{\mathbf{h}} \in T_{\mathbf{h}} \Big\} \cup P_{\mathbf{h}}, \quad (1)$$

where $T_{\mathbf{h}}$ denotes the set of all child histograms of $\mathbf{h}$ in T, and $P_{\mathbf{h}} = \emptyset$ if there exists a pair $(B_{\bar{\mathbf{h}}}, R_{\bar{\mathbf{h}}}) \in \mathcal{S}_{\bar{\mathbf{h}}}$ for some $\bar{\mathbf{h}} \in T_{\mathbf{h}}$ such that $\mathrm{pr}(R_{\bar{\mathbf{h}}}) \cap [p_{\mathbf{h}}, q_{\mathbf{h}}] \neq \emptyset$, and $P_{\mathbf{h}} = \big\{[p_{\mathbf{h}}, q_{\mathbf{h}}], [q_{\mathbf{h}}, q_{\mathbf{h}}]\big\}$ otherwise. All but $[q_{\mathbf{h}}, q_{\mathbf{h}}]$ intervals in $\mathcal{S}_{\mathbf{h}}$ are then valued 1, while the interval $[q_{\mathbf{h}}, q_{\mathbf{h}}]$, if it belongs to $\mathcal{S}_{\mathbf{h}}$, is valued 0.

The set $\mathcal{S}_{\mathbf{h}}$ can be computed in linear time, given $\mathcal{S}_{\bar{\mathbf{h}}}$ for all child histograms $\bar{\mathbf{h}}$ of $\mathbf{h}$ in T, by a pass over the boundary and using a stack data structure. Thus, the sets $\mathcal{S}_{\mathbf{h}}$ for each $\mathbf{h}$ in T can be computed in quadratic time.

Now, given the encoding $\mathcal{S}_{\mathbf{h}}$, we define a *realization* $\Gamma_{\mathbf{h}}$ of $\mathcal{S}_{\mathbf{h}}$ to be a set of intervals such that $\Gamma_{\mathbf{h}}$ contains at

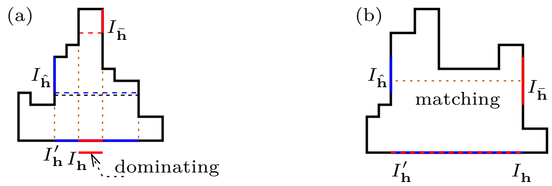

Figure 5: Illustrating domination and matching.

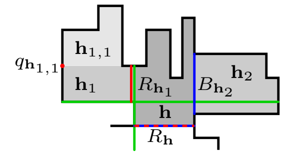

Figure 6: Illustrating the proof of Lemma 4. The green segments are the strip lighthouses in $\mathcal{L}(\Gamma_{\mathbf{h}})$.

most one interval from each interval pair in $\mathcal{S}_{\mathbf{h}}$. Each such realization $\Gamma_{\mathbf{h}}$ can be associated with the minimum canonical set $\mathcal{L}(\Gamma_{\mathbf{h}})$ of strip lighthouses such that each 1-valued interval in $\Gamma_{\mathbf{h}}$ is intersected by a lighthouse in $\mathcal{L}(\Gamma_{\mathbf{h}})$. Clearly, the number of possible realizations of $\mathcal{S}_{\mathbf{h}}$ is at most $3^{|\mathcal{S}_{\mathbf{h}}|}$. Note that all realizations do not necessarily correspond to parts of solutions to the SLP problem in $\mathbf{P}$.

Before we present the crucial recursive formula for the size of $\mathcal{L}(\Gamma_{\mathbf{h}})$, we need to introduce additional definitions. Let $I_{\mathbf{h}}$ and $I'_{\mathbf{h}}$ be two intervals on $b_{\mathbf{h}}$ such that $I_{\mathbf{h}} = \mathrm{pr}(I_{\bar{\mathbf{h}}})$ and $I'_{\mathbf{h}} = \mathrm{pr}(I_{\hat{\mathbf{h}}})$ for some $\bar{\mathbf{h}}, \hat{\mathbf{h}} \in T_{\mathbf{h}}$; see Figure 5(a). We say that $I_{\mathbf{h}}$ *dominates* $I'_{\mathbf{h}}$ if the subhistogram of $\mathbf{h}$ formed by cutting along the line segment with one endpoint at the lower endpoint of $I_{\hat{\mathbf{h}}}$ and the other endpoint by its horizontal projection in $\mathbf{h}$ contains $I_{\bar{\mathbf{h}}}$. If $I_{\mathbf{h}}$ is not dominated by any interval in a realization $\Gamma_{\mathbf{h}}$, we call $I_{\mathbf{h}}$ a *master* for $\Gamma_{\mathbf{h}}$. Next, we say that $I_{\mathbf{h}}$ and $I'_{\mathbf{h}}$ *match* if there is a point on $I_{\bar{\mathbf{h}}}$ whose horizontal projection in $\mathbf{h}$ lies on $I_{\hat{\mathbf{h}}}$; see Figure 5(b).

Now, let $V_{\Gamma_{\mathbf{h}}}$ denote the number of master intervals among the intervals in $\Gamma_{\mathbf{h}}$ and let $M_{\Gamma_{\mathbf{h}}}$ denote the number of distinctly matched intervals in $\Gamma_{\mathbf{h}}$, i.e., any interval is matched to at most one other interval. It is clear that $V_{\Gamma_{\mathbf{h}}}$ corresponds to the number of vertical strip lighthouses required in the realization $\Gamma_{\mathbf{h}}$, while $M_{\Gamma_{\mathbf{h}}}$ corresponds to the number of horizontal strip lighthouses that traverse $\mathbf{h}$ going from one child histogram of $\mathbf{h}$ to another. We thereby set

$$s(\Gamma_{\mathbf{h}}) = \begin{cases} 0 & \text{if } \mathbf{h} \text{ is a leaf and } \Gamma_{\mathbf{h}} = \{[q_{\mathbf{h}}, q_{\mathbf{h}}]\} \\ 1 & \text{if } \mathbf{h} \text{ is a leaf and } \Gamma_{\mathbf{h}} = \{[p_{\mathbf{h}}, q_{\mathbf{h}}]\} \\ V_{\Gamma_{\mathbf{h}}} - M_{\Gamma_{\mathbf{h}}} + \sum_{\bar{\mathbf{h}} \in T_{\mathbf{h}}} s(\Gamma_{\bar{\mathbf{h}}}) & \text{otherwise,} \end{cases}$$

which allows us to state the following lemma.

Lemma 4 *For every histogram $\mathbf{h}$, there is a realization $\Gamma_{\mathbf{h}}$ from $\mathcal{S}_{\mathbf{h}}$ such that $\mathcal{L}(\Gamma_{\mathbf{h}}) \cup \{b_{\mathbf{h}}\}$ is a solution to the SLP problem in $\mathbf{P}_{\mathbf{h}}$ and $s(\Gamma_{\mathbf{h}}) = |\mathcal{L}(\Gamma_{\mathbf{h}})|$, where $\mathbf{P}_{\mathbf{h}}$ is the subpolygon of $\mathbf{P}$ consisting of the histograms in T rooted at $\mathbf{h}$.*

Proof. Consider an optimal canonical solution $\mathcal{L}^*_{\mathbf{h}}$ to the LP problem in $\mathbf{P}_{\mathbf{h}}$. We prove that $\mathcal{L}^*_{\mathbf{h}}$ has a corresponding realization $\Gamma^*_{\mathbf{h}}$ of $\mathcal{S}_{\mathbf{h}}$ inductively in a bottom up fashion. We consider each subhistogram $\mathbf{h}_i$ in $T_{\mathbf{h}}$ separately.

If $\mathbf{h}_i$ is a leaf histogram in $T_{\mathbf{h}}$, then $\mathcal{L}^*_{\mathbf{h}}$ either has a lighthouse that intersects $[p_{\mathbf{h}_i}, q_{\mathbf{h}_i}]$ or there is a lighthouse in $\mathcal{L}^*_{\mathbf{h}}$ in the ancestor histogram of $\mathbf{h}_i$ for which the projection onto $b_{\mathbf{h}_i}$ intersects $q_{\mathbf{h}_i}$, otherwise not all of $\mathbf{h}_i$ is seen by $\mathcal{L}^*_{\mathbf{h}}$; see Figure 6 illustrating both these cases. Both these intervals $B_{\mathbf{h}_i} = [p_{\mathbf{h}_i}, q_{\mathbf{h}_i}]$ and $R_{\mathbf{h}_i} = [q_{\mathbf{h}_i}, q_{\mathbf{h}_i}]$ are paired in $\mathcal{S}_{\mathbf{h}}$ by construction and only one of them is used by $\mathcal{L}^*_{\mathbf{h}}$.

If $\mathbf{h}_i$ is an internal histogram in $T_{\mathbf{h}}$, then consider those histograms $\mathbf{h}_j$ that are children of $\mathbf{h}_i$ in $T_{\mathbf{h}_i}$. By induction, any (maximal) lighthouse in $\mathcal{L}^*_{\mathbf{h}}$ intersect the bases of $\mathbf{h}_j$ in at most one interval per pair lying in $\mathcal{S}_{\mathbf{h}_j}$ and furthermore these lighthouses intersect $\mathbf{h}_i$ between two (vertical) boundary edges (assuming $\mathbf{h}_i$ has our standard orientation). These two boundary edges define a (horizontal) interval and since $\mathcal{L}^*_{\mathbf{h}}$ is a solution to the SLP problem in $\mathbf{P}_{\mathbf{h}}$, it must have a lighthouse intersecting this interval. The interval is propagated from an interval in $\mathbf{h}_j$ and is therefore in an pair in $\mathcal{S}_{\mathbf{h}}$. The other interval from that pair is not propagated by induction, concluding the proof. □

The above lemma has the following immediate corollary.

Corollary 5 *There is a realization $\Gamma_{\mathbf{h}_t}$ from $\mathcal{S}_{\mathbf{h}_t}$ such that $\mathcal{L}(\Gamma_{\mathbf{h}_t}) \cup \{t\}$ is a solution to the SLP problem in $\mathbf{P}$ and $s(\Gamma_{\mathbf{h}_t}) = |\mathcal{L}(\Gamma_{\mathbf{h}_t})|$, where $\mathbf{h}_t$ is the root histogram in T.*

Clearly, an brute-force implementation of the above approach results in an exponential time algorithm for computing a solution to the SLP problem in a rectilinear polygon. However, in the following, we show that an "optimal" representation — and so an optimal set of strip lighthouses as well — can be (re-)constructed recursively, in a more efficient way, using some additional data when constructing the relevant sets $\mathcal{S}_{\mathbf{h}}$ of interval pairs.

For a given histogram $\mathbf{h}$, let $\Gamma^*_{\mathbf{h}}$ be an *optimal feasible* realization of $\mathcal{S}_{\mathbf{h}}$, i.e., a realization such that:

- $\mathcal{L}(\Gamma^*_{\mathbf{h}})$ has the smallest number of strip lighthouses among the realizations obtainable from $\mathcal{S}_{\mathbf{h}}$,
- the union of these lighthouses is a connected set,
- the whole of $\mathbf{P}_{\mathbf{h}}$ is covered.

It follows from Lemma 4 that $\mathcal{L}(\Gamma^*_{\bar{\mathbf{h}}}) \cup \{b_{\mathbf{h}}\}$ constitutes an optimal solution to the SLP problem in $\mathbf{P}_{\mathbf{h}}$. Note that $\mathcal{L}(\Gamma^*_{\mathbf{h}})$ is canonical in the sense defined above.

Lemma 6 *For every histogram* $\mathbf{h}$*, there is an optimal feasible realization* $\Gamma^*_{\mathbf{h}}$ *using only red intervals matched if possible, except when*

- *a red and a blue interval* (*but not the corresponding red intervals*) *match, or*
- *two blue intervals match* (*but not the corresponding red or red/blue intervals do*)*, at least one of which is a master,*

in which cases these matched intervals are used.

Proof. We sketch an inductive proof maintaining the following invariant: *the realization* $\Gamma^*_{\mathbf{h}}$ *induces as few strip lighthouses as possible and additionally the used intervals are as large as possible.*

If $\mathbf{h}$ is a leaf, the invariant holds trivially. On the other hand, if $\mathbf{h}$ is not a leaf, we assume without loss of generality, that considering intervals from two child histograms $\bar{\mathbf{h}}$ and $\hat{\mathbf{h}}$ in $T_{\mathbf{h}}$, then $\mathrm{pr}(R_{\bar{\mathbf{h}}}) \subset \mathrm{pr}(B_{\bar{\mathbf{h}}})$ and $\mathrm{pr}(R_{\hat{\mathbf{h}}}) \subset \mathrm{pr}(B_{\hat{\mathbf{h}}})$. We have four main cases.

1. The used interval $I_{\mathbf{h}}$ comes from the pair $P_{\mathbf{h}}$, see Equality (1).

   The argument holds trivially by the same consideration as for leaf histograms.

2. The used intervals $\mathrm{pr}(R_{\bar{\mathbf{h}}})$ and $\mathrm{pr}(R_{\hat{\mathbf{h}}})$ match with $R_{\bar{\mathbf{h}}}$ in $\Gamma^*_{\bar{\mathbf{h}}}$ and $R_{\hat{\mathbf{h}}}$ in $\Gamma^*_{\hat{\mathbf{h}}}$.

   Assume $\Gamma^*_{\mathbf{h}}$ does not contain both $\mathrm{pr}(R_{\bar{\mathbf{h}}})$ and $\mathrm{pr}(R_{\hat{\mathbf{h}}})$, then at least one of their corresponding blue intervals is used and by replacing it with the red interval, the size of the solution can be reduced by one contradicting optimality.

3. The used intervals $\mathrm{pr}(R_{\bar{\mathbf{h}}})$ and $\mathrm{pr}(B_{\hat{\mathbf{h}}})$ match with $R_{\bar{\mathbf{h}}}$ in $\Gamma^*_{\bar{\mathbf{h}}}$ and $R_{\hat{\mathbf{h}}}$ in $\Gamma^*_{\hat{\mathbf{h}}}$.

   We can assume that the red interval $\mathrm{pr}(R_{\hat{\mathbf{h}}})$ in the same pair as $\mathrm{pr}(B_{\hat{\mathbf{h}}})$ does not match $\mathrm{pr}(R_{\bar{\mathbf{h}}})$ (otherwise an optimal realization would have picked those two intervals, see Case 2). Assume $\Gamma^*_{\mathbf{h}}$ does not contain both of $\mathrm{pr}(R_{\bar{\mathbf{h}}})$ and $\mathrm{pr}(B_{\hat{\mathbf{h}}})$. If $\Gamma^*_{\mathbf{h}}$ contains $\mathrm{pr}(R_{\bar{\mathbf{h}}})$ and $\mathrm{pr}(R_{\hat{\mathbf{h}}})$ (not matching), it will use as many strip lighthouses but since $\mathrm{pr}(R_{\hat{\mathbf{h}}})$ is a subinterval of $\mathrm{pr}(B_{\hat{\mathbf{h}}})$, it violates the invariant; see Figure 7(a).

4. The used intervals $\mathrm{pr}(B_{\bar{\mathbf{h}}})$ and $\mathrm{pr}(B_{\hat{\mathbf{h}}})$ match and at least one of $\mathrm{pr}(R_{\bar{\mathbf{h}}})$ and $\mathrm{pr}(R_{\hat{\mathbf{h}}})$ is a master with $R_{\bar{\mathbf{h}}}$ in $\Gamma^*_{\bar{\mathbf{h}}}$ and $R_{\hat{\mathbf{h}}}$ in $\Gamma^*_{\hat{\mathbf{h}}}$.

   We can assume that the interval $\mathrm{pr}(R_{\bar{\mathbf{h}}})$ does not match $\mathrm{pr}(B_{\hat{\mathbf{h}}})$ and that $\mathrm{pr}(B_{\bar{\mathbf{h}}})$ does not match $\mathrm{pr}(R_{\hat{\mathbf{h}}})$ (otherwise an optimal realization would have picked one of those two pairs of matching intervals, see Case 3). Assume $\Gamma^*_{\mathbf{h}}$ contains both $\mathrm{pr}(R_{\bar{\mathbf{h}}})$ and $\mathrm{pr}(R_{\hat{\mathbf{h}}})$, then $\mathbf{h}$ must contain two horizontal strip lighthouses (assuming the standard orientation of $\mathbf{h}$), one intersecting each of $R_{\bar{\mathbf{h}}}$ and $R_{\hat{\mathbf{h}}}$, and two vertical strip lighthouses, one intersecting each of the horizontal ones. However, this solution violates the invariant since it can be replaced by one horizontal strip lighthouse intersecting both $B_{\bar{\mathbf{h}}}$ and $B_{\hat{\mathbf{h}}}$, since $\mathrm{pr}(B_{\bar{\mathbf{h}}})$ and $\mathrm{pr}(B_{\hat{\mathbf{h}}})$ match, and one vertical strip lighthouse in $\mathbf{h}$ intersecting the horizontal one. The size of the solution remains the same but the interval $\mathrm{pr}(B_{\bar{\mathbf{h}}}) = \mathrm{pr}(B_{\hat{\mathbf{h}}})$ contains each of $\mathrm{pr}(R_{\bar{\mathbf{h}}})$ and $\mathrm{pr}(R_{\hat{\mathbf{h}}})$; see Figure 7(b).

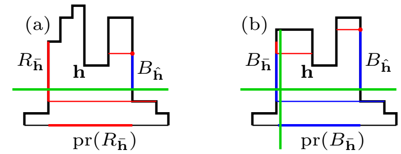

Figure 7: Illustrating the proof of Lemma 6. The green segments are the strip lighthouses.

Enumerating the remaining possibilities, it follows that in these cases the invariant holds by always selecting and propagating the red intervals. □

The algorithm makes passes over the boundary of the histogram and computes, given $\mathcal{S}_{\bar{\mathbf{h}}}$ for each child histogram $\bar{\mathbf{h}}$, a maximal set of unique red/red matching intervals, given these, a maximal set of unique red/blue matching intervals, and given these, a maximal set of unique blue/blue matching intervals where at least one is a master. After the matching intervals have been established the algorithm selects the remaining red ones for an optimal solution.

It is easy to verify that the running time of the above algorithm is quadratic (with respect to the number of vertices of the input polygon $\mathbf{P}$).

Theorem 7 *A solution to the SLP problem for a simple rectilinear polygon* $\mathbf{P}$ *and a target edge* t *can be computed in quadratic time.*

For rectilinear polygons with holes, a simple modification of the NP-hardness proof for optimally guarding a rectilinear polygon with holes [8], adding a target notch in a corner of the construction shows the result.

Theorem 8 *The SLP problem in rectilinear polygons with holes is NP-hard.*

3.1 Edge Lighthouses

Another natural model is to assume that an edge lighthouse sees all points of the histogram with this edge as its base. Unfortunately, our standard navigation protocol can get stuck in this model and therefore another navigation protocol is needed: *If stuck at a lighthouse with the identifying number l, then move towards the lighthouse with the maximum number you see until you see a lighthouse with an identifying number smaller than l (and then continue with the standard protocol until you either reach the target or get stuck again)*. This modification is sufficient (for the existence of a solution), since by taking all edges as lighthouses and labelling them consecutively along the boundary (with 0 at the target, and then increasingly in counterclockwise manner), we obtain a feasible solution.

4 Laser Lighthouses in Grids

In grids, a natural counterpart of (directed) edge lighthouses in rectilinear polygons, are laser lighthouses. Specifically, a *laser lighthouse* is a point which can illuminate only towards one of the forth directions: North, East, South or West. We begin with two simple but crucial observations.

Observation 1 *For any grid, n is a lower bound.*

Observation 2 *Putting a laser not at the endpoint of a segment results in two lasers associated with this segment, so in order to have only n lasers, each segment must have at least one endpoint being an intersection point and all of the lasers must be located at some of these intersection end-points.*

Taking into account the above two remarks, one can easily observe that $2n$ is an upper bound. The idea is to place two lasers on each grid segment, starting from the (at most) two segments that the target is located at (and so, at most four lasers in total), in opposite direction, and continue with each new, so far uncovered segment intersecting some covered one, by placing another two lasers, at the intersection point, in the directions alternative to the intersected already covered segment. Therefore, we have a simple linear-time 2-approximation algorithm.

In general, the problem is polynomially tractable; recall that by Observation 1, each segment must be assigned at least one laser. The idea of our algorithm is as follows. For a grid $\mathbf{G} = V_{\mathbf{G}} \cup H_{\mathbf{G}}$, where $V_{\mathbf{G}}$ ($H_{\mathbf{G}}$, resp.) is the set of all vertical (horizontal, resp.) line segments of $\mathbf{G}$, we first construct the weighted directed bipartite intersection graph $G_{\mathbf{G}}$, with the bipartition $(V_{\mathbf{G}}, H_{\mathbf{G}})$ [17], where the weight $w(a)$ of an arc $a = (x, y)$, corresponding to the intersection of line segments x and y, is set to 1 if x has an endpoint on y, and 2 otherwise. (The graph $G_{\mathbf{G}}$ can be constructed in $O(n \log n + m)$ time, where $m = O(n^2)$ is the number of intersection points of grid segments of $\mathbf{G}$ [2].) Then we modify the graph $G_{\mathbf{G}}$by adding the new vertex t that corresponds to the target point t, and by adding at most two arcs (z, t), depending on the location of t on a line segment z, with the weight of (z, t) equal to 1 if t is located at the endpoint of z, and 2 otherwise.

Let $D_{\mathbf{G}}$ be the resulting digraph from the above modification of $G_{\mathbf{G}}$. One can that observe that any arborescence T_t of $D_{\mathbf{G}}$ with the root t corresponds to a feasible laser assignment in $\mathbf{G}$ with the number of used lasers equal to the cost of T_t, and *vice versa*. Therefore, there is one-to-one correspondence between any arborescence of $D_{\mathbf{G}}$ with the root at t and an optimal solution to the Lighthouse Problem in $\mathbf{G}$. Consequently, since the problem of computing a minimum spanning tree of a weighted digraph can be solved in $O(n \log n + m)$ time [10], where n and m are the number of vertices and edges of the input graph, respectively, we immediately obtain the following result.

Theorem 9 *The Laser Lighthouse Problem in grids can be solved in $O(n \log n + k)$ time, where n is the number of grid segments of the input grid while k is the number of their intersection points.*

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